Forchheimer flow in gently sloping layers: application to drainage of porous asphalt

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Abstract

This paper presents analytical solutions for the problem of steady one-dimensional Forchheimer flow in an unconfined layer. The study's motivation is the drainage behavior of a highway pavement called permeable friction course. Permeable friction course is a layer of porous asphalt placed on top of impermeable pavement. Porous overlays are growing in popularity because they reduce noise, mitigate hazards of wet weather driving, and produce cleaner runoff. Several of these benefits occur because water drains within the pavement rather than on the road surface. Drainage from the friction course is essentially that of an unconfined aquifer and has been successfully modeled using Darcy's law and the Dupuit-Forchheimer assumptions. Under certain cases, drainage may occur outside of the range where Darcy's law applies. The purpose of this paper is to identify cases where the assumption of Darcy flow is violated, develop analytical solutions based on Forchheimer's equation, and compare the solutions with those obtained for the Darcy case. The principle assumptions used in this analysis are that the relationship between hydraulic gradient and specific discharge is quadratic in nature (Forchheimer's equation) and that the Dupuit-Forchheimer assumptions apply. Comparing the Darcy and Forchheimer solutions leads to a new criterion for assessing the applicability Darcy's law termed the discharge ratio.

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1 Introduction

Seepage through coarse material occurs in earthen embankments, gravel deposits, landfill caps, and porous pavements. Drainage through porous pavements has received extensive study in recent years as the benefits of such pavements become clearer and they receive wider use. In particular, an overlay of porous asphalt known as permeable friction course (PFC) or open graded friction course has been shown to have numerous benefits in the highway environment. PFC is composed of coarse aggregates, fine aggregates, asphalt binders and stabilizing additives; it is placed in a 20-50mm thick layer on top of regular impermeable pavement (*NCHRP*, 2009). Most installations have a maximum aggregate size of 19mm and an initial porosity of 18-22% (*NCHRP*, 2009). Reported values of hydraulic conductivity are on the order of 1 cm/s (*Charbeneau et al.*, 2011).

Compared to conventional asphalt or concrete pavement, a porous overlay is rougher and so provides better traction both in wet and dry conditions (*NCHRP*, 2009). Noise measurements show that an overlay is quieter than conventional pavement and this is the most common basis for installation (*NCHRP*, 2009). Monitoring has shown that the runoff from overlays is much cleaner than conventional pavement (*Stotz and Krauth*, 1994; *Berbee et al.*, 1999; *Pagotto et al.*, 2000; *Barrett*, 2008; *Eck et al.*, 2012b). And finally, porous overlays improve driving conditions in wet weather by reducing splash and spray and improving visibility (*NCHRP*, 2009).

The benefits of cleaner runoff, reduced splash and spray and improved visibility are directly related to the hydraulic performance of the porous layer. These benefits occur because the pavement facilitates removal of water from the road surface. In order to realize and assure these benefits, the hydraulic performance of the overlays is of interest. Conceptually, the problem is that of unconfined seepage through an inclined porous layer under areal recharge. The solution is the position of the piezometric surface within the porous layer. If the inclination of the porous layer is constant then the problem is one-dimensional. If the recharge rate is constant, the system reaches a steady state. If both the inclination and recharge are constant, the saturated thickness varies only along the direction of the inclination (drainage path) according to a first order ordinary differential equation (ODE).

The steady state 1D problem has been studied for the case of a linear relationship between hydraulic gradient and specific discharge (Darcy's law). *Ranieri* (2002) provided the first predictions of water depth within PFC layers through a numerical solution to the steady state ODE. *Charbeneau and Barrett* (2008) provided analytical solutions to the problem and found good agreement with the results of *Ranieri*. *Tan et al.* (2004) provide numerical results from a commercial finite element package that differ from the other workers, though the exact source of the differences is unclear.

Darcy's law has also been applied to the 2D unsteady problem by *Eck et al.* (2012a), who also examine the coupling between seepage within the porous layer and sheet flow on top of it. An analytical treatment for the coupled surface/subsurface case is given by *Eck et al.* (2011).

All of these studies require some information about the hydraulic properties of the PFC layer and these have been studied by *Ranieri* (2002), *Charbeneau et al.* (2011), and *Klenzendorf* (2010). The studies share the common finding that a non-linear relationship is observed between hydraulic gradient and specific discharge.

In porous media flows, a linear relationship between the specific discharge and hydraulic gradient is known as Darcy's law (*Darcy*, 1856):

$$q = -KI \tag{1}$$

where q is the specific discharge or Darcy velocity, I is the hydraulic gradient, and K is the hydraulic conductivity. Darcy's law applies when microscopic velocities are slow enough that convective accelerations in the momentum equation may be ignored; and, relatedly that the flow is laminar. These assumptions suggest three regimes for porous media flow which are outlined by *Bear* (1972): (1) a linear regime where Darcy's law applies; (2) an inertial regime where convective accelerations play a role, but the flow remains laminar; and, (3) a fully turbulent regime.

A porous media Reynolds number is used to distinguish between the regimes.

$$Re_p = \frac{dq}{\nu} \tag{2}$$

where d is the mean grain diameter; ν is the kinematic viscosity; and q is the specific discharge. Suggested upper limits for the linear and inertial regimes are $Re_p < 1$ *to* 10 and $Re_p = 100$, respectively (*Bear*, 1972).

Outside the linear flow regime, the relationship between hydraulic gradient and specific discharge is non-linear and usually described by either a power law or a polynomial equation. Power law forms similar to $I = -Kq^n$ are known as Izbash's law (*Izbash*, 1931). The polynomial form is known as Forchheimer's equation (*Forchheimer*, 1901) and may be written in terms of the hydraulic gradient as

$$-I = \alpha q + \beta q^2 \tag{3}$$

where α and β are coefficients, I is the hydraulic gradient, and q is the specific discharge. Following the convention of *Ruth and Ma* (1992), β is referred to as the Forchheimer coefficient. Forchheimer's equation and the Izbash law are equivalent in terms of representing measured relationships between hydraulic gradient and specific discharge, but the Forchheimer formulation is more theoretically sound (*Bordier and Zimmer*, 2000).

Comparing Forchheimer's equation (3) to Darcy's law (Eq. 1), shows that the equations are equivalent when $\beta = 0$ and $\alpha = 1/K$. This comparison also shows that Forchheimer's equation requires a higher hydraulic gradient to carry the same specific discharge owing to the quadratic contribution of inertial effects; inertial effects increase the overall resistance to flow. In the case of an unconfined layer, the result of this increased resistance is a greater piezometric height. Therefore, Darcy's law under-predicts the water depth in highly permeable porous layers. Selecting the thickness of a PFC layer using Darcy's law rather than Forchheimer's equation will result in a thinner layer and more surface discharge.

Analytical solutions for unconfined inertial flows in porous media include *Basak and Madhav* (1979) who consider transient non-Darcy drainage through trapezoidal embankments. Their solution requires use of a semi-empirical function that is calibrated using model studies. *Bordier and Zimmer* (2000) use the Izbash formulation and find a steady state analytical solution for a flat aquifer and provide numerical solutions for several transient cases. *Moutsopoulos* (2007) investigates transient flows induced by a sudden rise in the water table. He considers cases where inertial effects associated with the quadratic part of Forchheimer's equation dominate and neglects the linear portion of the equation.

The primary contribution of the present paper is a set of analytical solutions for Forchheimer seepage in an unconfined sloping aquifer. The solutions are of interest to hydrologists, designers of porous pavement highways and landfill caps, and modelers seeking to validate more comprehensive numerical treatments. Applying the analytical solutions requires an initial point or boundary condition and an estimate of the Forchheimer coefficient. For the boundary condition, a physically based approach is suggested. For the Forchheimer coefficient in porous asphalts, a correlation with the hydraulic conductivity based on recent experimental measurements conducted by others is provided. Finally, the specific discharge under the assumptions of Darcy and Forchheimer flow is compared by ratio. Contour plots of this discharge ratio allow for clear and fast estimates of the effect of assuming linear flow.

2 Analytical Solutions for Forchheimer Flow

A schematic view of seepage through an unconfined porous layer is shown in Fig 1. The elevation of the bottom of the porous layer with respect to a datum is z. The thickness of the piezometric surface within the layer is h; the hydraulic head is H; the flow rate per unit width is U; and the constant rainfall or recharge rate is r. The origin of the coordinate system is fixed at x = 0, a no flow boundary.

The flow rate per unit width is the integral of the specific discharge over the saturated thickness. From continuity, this quantity also equals the product of the rainfall rate and horizontal coordinate.

$$U = \int_0^h q \, dh = rx \tag{4}$$

Since q depends on $\frac{dH}{dx}$, but not on h directly, integration gives $qh = rx$. The assumptions that q and $\frac{dH}{dx}$ are proportional and that pressure is hydrostatic are those of Dupuit-Forchheimer. For one-dimensional flows, the Dupuit-Forchheimer assumptions give the correct discharge and errors in the hydraulic head on the order of 0.25% *Irmay* (1967). Solving for q, substituting into Forchheimer's equation, and writing $1/K$ instead of α gives

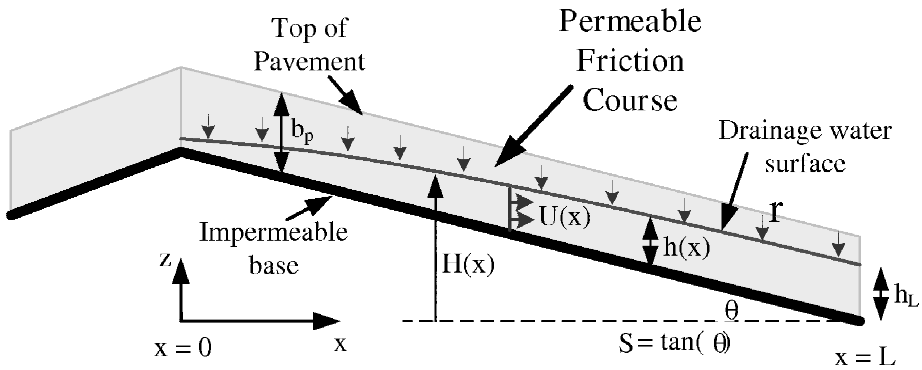

Figure 1: Schematic view of drainage in a porous layer (adapted from *Charbeneau and Barrett* (2008), with permission of AGU)

$$I = -\frac{\beta r^2 x^2}{h^2} - \frac{rx}{Kh} \tag{5}$$

K has been retained because it is a more common characteristic of a porous medium than α, though either could be used for the derivations that follow. The hydraulic gradient is composed of the depth gradient and the inclination of the aquifer (S) according to

$$I = S + \frac{dh}{dx} \tag{6}$$

Substituting Eq. 6 into Eq. 5 yields the ordinary differential equation that governs steady, Forchheimer flow in an inclined aquifer

$$\frac{dh}{dx} = -\frac{\beta r^2 x^2}{h^2} - \frac{rx}{Kh} - S \tag{7}$$

The individual terms of Eq. 7 are dimensionless, showing that three essential parameters govern the problem:

1. The inclination of the aquifer, S;
2. The recharge or rainfall rate relative to the hydraulic conductivity, $\frac{r}{K}$; and,
3. The product of the Forchheimer coefficient and the square of the rainfall, βr^2.

Setting $\beta = 0$ in Eq. 7 reduces the number of parameters and returns the Darcy flow form given by *Charbeneau and Barrett* (2008) (Eq. 7). Eq. 7 of the present paper may be separated by the following change of variables.

$$\eta = \frac{h}{x} \tag{8}$$

$$\frac{dh}{dx} = x\frac{d\eta}{dx} + \eta \tag{9}$$

Making the transformation and defining the dimensionless variables R_1 and R_2 we have

$$-\frac{dx}{x} = \frac{\eta^2 d\eta}{\eta^3 + S\eta^2 + R_1\eta + R_2} \tag{10}$$

where

$$R_1 = \frac{r}{K} \tag{11}$$

$$R_2 = \beta r^2 \tag{12}$$

Comparison with the transformed ODE for the Darcy case (see *Charbeneau and Barrett* (2008), Eq. 11) shows that the effect of using the Forchheimer equation is to increase the order of the polynomial in the denominator of the ODE.

The integral of Eq. 10 depends on the denominator of the right hand side, which in turn depends on the discriminant of the cubic polynomial. A discriminant for cubic polynomials is given by *Abramowitz and Stegun* (1965) and is equivalent to:

$$\Delta = P^2 - Q^3 \tag{13}$$

where P and Q are given by *Press et al.* (1992) as:

$$P = \frac{2S^3 - 9SR_1 + 27R_2}{54} \tag{14}$$

$$Q = \frac{S^2 - 3R_1}{9} \tag{15}$$

The number and type of roots to the cubic polynomial varies according to the sign of the discriminant:

1. $\Delta < 0$ gives three real roots;
2. $\Delta = 0$ gives three real roots including a double root; and,
3. $\Delta > 0$ gives one real and two complex conjugate roots.

Other formulations of the discriminant for cubic polynomials appear in the literature (e.g. *Bartsch* (1974)) and the sign of the discriminant as related to the number of real roots varies by formulation. Although the numerical value of the discriminants can differ, the result applied to any equation is the same.

Eq. 10 is a separable, non-linear ordinary differential equation of first order. Integration gives a family of solutions that differ by a constant. This constant may be evaluated using any initial point on the solution curve. The point $h(x = L) = h_L$ lies at the boundary of the domain of Eq. 10 and is a convenient choice of initial point.

The transformed and separated ODE (Eq. 10) is not directly integrable due to the cubic denominator, but may be factored depending on the number of real roots. The following subsections address each of the three possibilities for the discriminant. Each case gives a different factorization of Eq. 10, which is integrated to obtain a general solution. The constant of integration is obtained by evaluating the point $h(x = L) = h_L$ and the resulting solution is given as an implicit function.

2.1 Case I

Case I corresponds to low rainfall intensities or, equivalently, high drainage capacity of the porous layer. This physical interpretation is explained further in section 5. With three real roots, Eq. 10 becomes

$$-\frac{dx}{x} = \frac{\eta^2\, d\eta}{(\eta - \lambda_1)(\eta - \lambda_2)(\eta - \lambda_3)} \tag{16}$$

where λ_1, λ_2 and λ_3 are the real roots of the cubic and are found from the values of P, Q and the parameter S using formulas given by *Press et al.* (1992):

$$\lambda_1 = -2\sqrt{Q}\, cos\frac{\varphi}{3} - \frac{S}{3} \tag{17}$$

$$\lambda_2 = -2\sqrt{Q}\, cos\left(\frac{\varphi + 2\pi}{3}\right) - \frac{S}{3} \tag{18}$$

$$\lambda_3 = -2\sqrt{Q}\, cos\left(\frac{\varphi - 2\pi}{3}\right) - \frac{S}{3} \tag{19}$$

where φ is calculated from

$$\varphi = arccos\left(\frac{P}{\sqrt{Q^3}}\right) \tag{20}$$

The three essential parameters of the problem (S, R_1, R_2) are contained within the three real roots. Eq. 16 may be decomposed using partial fractions and integrated. Using the point $h(x = L) = h_L$ to evaluate the integration constant and re-arranging gives the solution in implicit form:

$$\begin{aligned} FF_1(x, h; \lambda_1, \lambda_2, \lambda_3) \;=\; & ln\left(\frac{x}{L}\right) \\ & + A\, ln\left(\frac{h/x - \lambda_1}{h_L/L - \lambda_1}\right) \\ & + B\, ln\left(\frac{h/x - \lambda_2}{h_L/L - \lambda_2}\right) \\ & + C\, ln\left(\frac{h/x - \lambda_3}{h_L/L - \lambda_3}\right) = 0 \end{aligned} \tag{21}$$

where FF_1 is the implicit solution function for Case I Forchheimer flow, and $A, B, \&C$ are constants from the partial fraction decomposition that are combinations of the roots of the cubic.

$$A = \frac{\lambda_1^2}{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)} \tag{22}$$

$$B = \frac{\lambda_2^2}{(\lambda_2 - \lambda_1)(\lambda_2 - \lambda_3)} \tag{23}$$

$$C = \frac{\lambda_3^2}{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)} \tag{24}$$

In Eq. 21, values of h are restricted to maintain positive arguments within the logarithms. If $\lambda < 0$ then both the numerator and denominator will be positive since $\frac{h}{x} \geq 0$ and the only restriction is $h \geq 0$. For $\lambda > 0$ the sign of denominator depends on the magnitude of $\frac{h_L}{L}$ compared to λ. If $\lambda > \frac{h_L}{L}$, λ is large and positive; the denominator is negative and numerator must satisfy $\frac{h}{x} - \lambda < 0$, or equivalently $h < x\lambda$. If instead $\lambda < \frac{h_L}{L}$, λ is small and positive; the denominator is positive and the numerator must satisfy $\frac{h}{x} - \lambda > 0$ or equivalently $h > x\lambda$. In this way, each root of the cubic implies a restriction on acceptable values of h. The restriction corresponding to each root should be calculated with the most restrictive range for h taken.

The drainage profile $h(x)$ is obtained by finding the roots of Eq. 21. The limiting values of h may be used to bracket the root. Many root finding methods are available. The method of bi-section was chosen for ease of implementation.

An example of a drainage profile corresponding to this case is shown in Fig. 2 along with the Darcy flow solution of *Charbeneau and Barrett* (2008) and a numerical solution obtained using the Runge-Kutta method. Both the hydraulic profile and depth profile are shown to illustrate the difference between the Forchheimer and Darcy solutions. As expected, the additional flow resistance due to inertial effects increases the depth for the Forchheimer case. The maximum absolute error of the Darcy solution with respect to the Forchheimer one is -0.377 cm near $x = 750$cm and corresponds to -12.7%. The maximum relative error occurs as $h \to 0$ approaching $x = 0$.

Although bi-section is easily implemented, care is required to extract the roots. The limits on h found above are asymptotes of the function FF_1 and the root is sometimes very near the asymptote (Fig. 3). The bi-section algorithm should test for a sign change across the brackets of the root; however, since the initial brackets are asymptotes they cannot be tested directly. The limiting values of h should be incremented by the tolerance of the algorithm to check for sign change. Small tolerances $\epsilon < 10^{-12}$ may be needed approaching $x = 0$ in order to extract the root.

2.2 Case II

Case II is a special case of limited practical interest. It corresponds to a type of critical flow in which the rainfall rate is just balanced by the slope and drainage

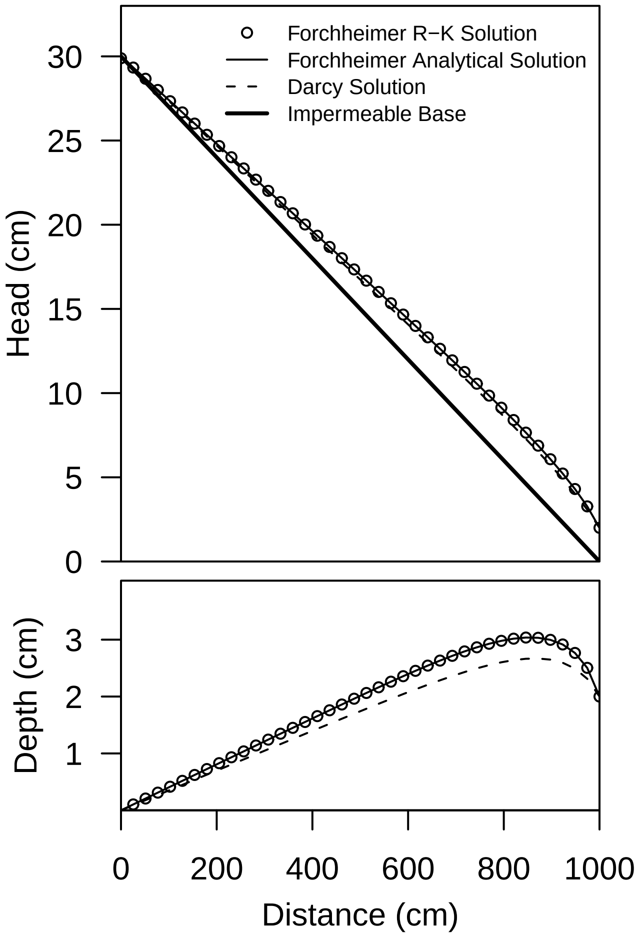

Figure 2: Comparison of drainage profiles computed using Forchheimer's equation and Darcy's law for rainfall at 1cm/hr, slope of -3%, hydraulic conductivity 3 cm/s, $\beta = 0.64 s^2/cm^2$, and $h_L = 2cm$. R-K refers to 4th order Runge-Kutta integration using 40 steps. Discriminant is negative for this case as shown in Fig. 7(c).

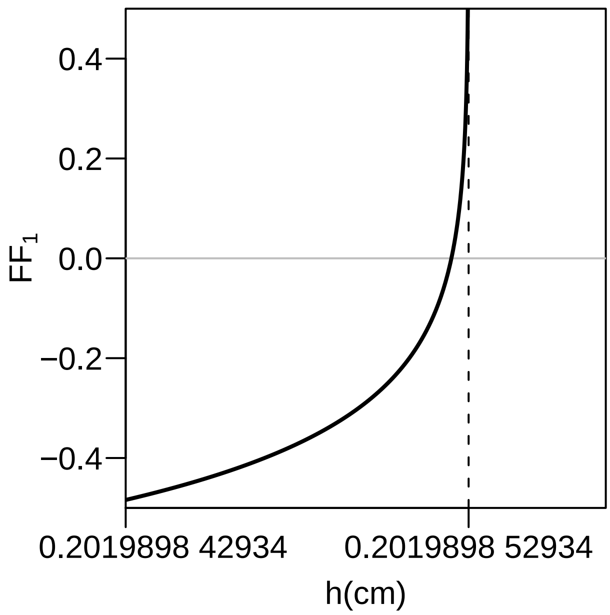

Figure 3: Variation in the implicit solution function for Case 1 (low rainfall rates) when x = 50cm and other parameters are the same as Fig. 2. The root lies $5.0277 \cdot 10^{-10} cm$ from the asymptote.

capacity of the pavement. All of the roots are real, but one root appears twice. The ODE may be written

$$-\frac{dx}{x} = \frac{\eta^2 \, d\eta}{(\eta - \lambda_1)(\eta - \lambda_2)^2} \tag{25}$$

where λ_1 is the single root and λ_2 is the double root. The roots are found from Eqs. 17 - 20. After partial fraction decomposition, integration, evaluation of the integration constant and rearrangement, the solution function is:

$$\begin{aligned} FF_2(x, h; \lambda_1, \lambda_2, h_L, L) &= ln\left(\frac{x}{L}\right) \\ &\quad + A\, ln\left(\frac{h/x - \lambda_1}{h_L/L - \lambda_1}\right) \\ &\quad + B\left(\frac{1}{h_L/L - \lambda_2} - \frac{1}{h/x - \lambda_2}\right) \\ &\quad + C\, ln\left(\frac{h/x - \lambda_2}{h_L/L - \lambda_2}\right) = 0 \end{aligned} \tag{26}$$

where A, B, and C are combinations of λ_1 and λ_2

$$A = \frac{\lambda_1^2}{(\lambda_1 - \lambda_2)^2} \tag{27}$$

$$B = \frac{\lambda_2^2}{(\lambda_2 - \lambda_1)^2} \tag{28}$$

$$C = \frac{B\lambda_1 - A\lambda_2^2}{\lambda_1 \lambda_2} \tag{29}$$

The appearance of logarithmic terms in the solution function suggests the same restrictions on h as in Case I. These restrictions can be used to bracket the appropriate root as described previously.

2.3 Case III

Case III corresponds to high rainfall rates and low drainage capacity of the porous layer. The cubic has one real root, which is used to factor the denominator of Eq. 10 into linear and quadratic terms. The factorization gives:

$$-\frac{dx}{x} = \frac{\eta^2 \, d\eta}{(\eta - \lambda_1)(\eta^2 + B\eta + C)} \tag{30}$$

The coefficients of the quadratic part are found by polynomial long division (*Weisstein*). The coefficients contain the real root and two of the three essential parameters so that in total three parameters still govern the problem.

$$B = S + \lambda_1 \tag{31}$$

$$C = R_1 + \lambda_1 B \tag{32}$$

In this case, λ_1 is computed from (*Press et al.*, 1992)

$$\lambda_1 = (M + \frac{Q}{M}) - \frac{S}{3} \tag{33}$$

where

$$M = -sgn(P)\left[|P| + \sqrt{P^2 - Q^3}\right]^{1/3} \tag{34}$$

and P and Q are from Eqs. 14 - 15.

Integration proceeds from a partial fraction decomposition, followed by completing the square in the quadratic term and re-arranging. The integration constant is evaluated from $h(x = L) = h_L$, and the solution function is:

$$\begin{aligned} FF_3\left(x, h; \lambda_1, h_L, L\right) &= ln\left(\frac{x}{L}\right) \\ &+ D \cdot ln\left(\frac{h/x - \lambda_1}{h_L/L - \lambda_1}\right) \\ &+ \frac{E}{2} \cdot ln\left[\frac{\left(\frac{h}{x}\right)^2 + B\,\frac{h}{x} + C}{\left(\frac{h_L}{L}\right)^2 + B\,\frac{h_L}{L} + C}\right] \\ &+ \frac{G}{\sqrt{C - \frac{B^2}{4}}}\left[arctan\left(\frac{h/x + B/2}{\sqrt{C - \frac{B^2}{4}}}\right) - arctan\left(\frac{h_L/L + B/2}{\sqrt{C - \frac{B^2}{4}}}\right)\right] \end{aligned} \tag{35}$$

where D, E, F, and G are constants from the decomposition and re-arrangement.

$$D = \frac{\lambda_1}{B + \lambda_1 + C/\lambda_1} \tag{36}$$

$$E = 1 - D \tag{37}$$

$$F = \frac{D \cdot C}{\lambda_1} \tag{38}$$

$$G = F - \frac{B \cdot E}{2} \tag{39}$$

As in the previous two cases, the value of λ_1 restricts the acceptable values of h. When λ_1 is negative $h > 0$ and the upper bracket for h may be found through an iterative procedure as follows. An estimate for h at location x can be found from the problem parameters by setting $\frac{dh}{dx} = 0$ in Eq. 7. Solving the quadratic in h yields a starting value for the upper bound:

$$h = \frac{-R_1 x - \sqrt{(R_1 x)^2 - 4SR_2x^2}}{2S} \tag{40}$$

The resulting value of h tested in FF_3 is incremented by h_L until the sign of $FF_3(h, x)$ differs from the sign of $FF_3(0, x)$. Once the root is bracketed it can be found by bi-section or another algorithm.

A drainage profile corresponding to Case III is shown in Fig. 4. This example uses the same parameters as the Case I example, but a higher rainfall rate of 5cm/hr. The maximum absolute error of the Darcy solution with respect to the Forchheimer one is 1.74 cm near $x = 770$cm. The maximum relative error of -30.0% occurred at $x = 0$ cm. The solution at $x = 0$ is found by polynomial interpolation as described in *Charbeneau and Barrett* (2008). The high gradient caused by the value of h_L also increases the error of the numerical solution.

3 Choosing the value of h_L

From a mathematical point of view, any point along the solution curve can be used to evaluate the constant of integration. The point $h(x = L) = h_L$ is chosen because this point is at the boundary of the region over which the ODE applies. The approach to choosing h_L depends on the possible presence of sheet flow.

3.1 h_L with Sheet Flow

The development of the governing equation and solutions thereto has assumed that the porous layer has infinite thickness but this is not true in practice. All porous layers have finite thickness and porous asphalt overlays have a thickness of a few centimeters. When the volume of drainage exceeds the capacity of the porous layer, discharge occurs as sheet flow.

Once sheet flow begins the hydraulic gradient is limited to the pavement slope and the flow depth is equal to the pavement thickness. Fixing these quantities also fixes the discharge and so the problem becomes finding the point where this discharge occurs. The value is found by setting $I = S$ and $h = b$ in Eq. 5. This gives a quadratic equation in x. In applying the quadratic formula, the positive root is appropriate because the distance L is positive.

$$L_{sheet} = \frac{-R_1/b + \sqrt{(R_1/b)^2 - 4SR_2/b^2}}{2R_2/b^2} \tag{41}$$

When $L_{sheet} < L$, sheet flow occurs and the integration constant is evaluated using the point $h(x = L_{sheet}) = b$. As an example, consider a porous layer 5cm thick with hydraulic conductivity of 2 cm/s, a drainage slope of 3%, a rainfall rate of 1.5cm/hr, and a no-flow boundary at the upstream end. Under Darcy flow, sheet flow occurs 720 cm from the no flow boundary. Under Forchheimer flow, and assuming $\beta = 0.984 s^2/cm^2$ from Eq. 42, sheet flow begins 650 cm from the no-flow boundary (Fig. 5). The basis for Eq. 42 and methods for choosing the Forchheimer coefficient are discussed in Section 4.

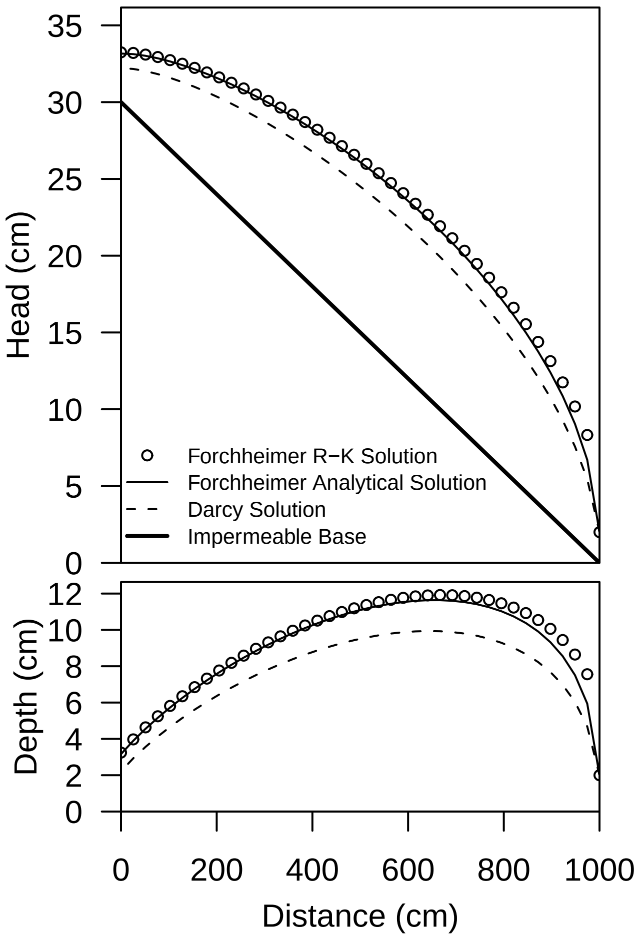

Figure 4: Comparison of drainage profiles computed using Forchheimer's equation and Darcy's law for rainfall at 5cm/hr, slope of -3%, hydraulic conductivity 3 cm/s, and $\beta = 0.64 s^2/cm^2$, and $h_L = 2cm$. R-K refers to 4th order Runge-Kutta integration using 40 steps.

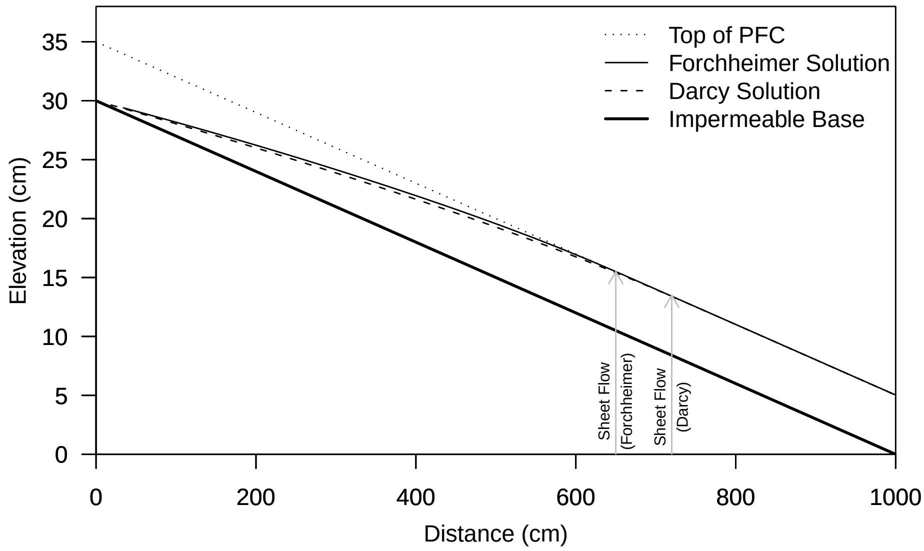

Figure 5: Drainage profiles for a 5cm porous layer with sheet flow under Forchheimer (solid) and Darcy (dashed) flow. Parameters were $K = 2cm/s$, $\beta = 0.984s^2/cm^2$ by power fit, and rainfall of 1.5 cm/hr.

3.2 h_L without Sheet Flow

When L_{sheet} as computed above falls beyond the length of the porous layer, at least two approaches to choosing h_L are possible. The preferred approach for evaluating the drainage of porous overlays is to use the location of sheet flow as found above even though it lies beyond the length of the porous layer. As an alternative, an arbitrary value may be assigned to h_L as was done in Figs. 2 and 4. This approach applies when the porous layer discharges into a water body with a known surface elevation.

4 Choosing the Forchheimer Coefficient

An essential step in applying the Forchheimer equation is selecting a value for the Forchheimer coefficient. Numerous relationships (e.g. *Ward* (1964), *Ergun* (1952), *Kadlec and Knight* (1996) among others) exist in the literature to predict both the hydraulic conductivity and Forchheimer coefficient based on the properties of the porous medium and the fluid; the performance of these equations has been evaluated by *Sidiropoulou et al.* (2007).

Ideally the value of the Forchheimer coefficient for the medium of interest is found by experiment. In the case of porous asphalt overlays, recent experiments by *Charbeneau et al.* (2011) and, relatedly, *Klenzendorf* (2010) have used a combination of numerical analysis and laboratory experiments to determine the hydraulic conductivity and Forchheimer coefficient for porous asphalt overlays. Tests were made on PFC core samples using a radial flow permeameter. Core samples were sandwiched between a solid bottom plate and a top plate fitted with a stand pipe. A constant flow of water entered the core vertically through the standpipe and exited radially at the circumference of the submerged core. The hydraulic conductivity and Forchheimer coefficient for each core were computed from a series of tests using different flow rates and head differences as described in the references. The resulting values of the Forchheimer coefficient as a function the hydraulic conductivity are plotted in Fig. 6. A relationship between β and K is expected because both reflect the character of the porous medium.

To have a parametric expression for the Forchheimer coefficient in PFC layers, a power–law fit to Klenzendorf's measurements is provided. The regression computations were performed using the R environment (*R Development Core Team*, 2010).

$$\beta = e^{0.710091} K^{-1.04806} \tag{42}$$

The equation has 28 degrees of freedom, a residual standard error of 0.5601 s^2/cm^2 and an adjusted R-squared of 0.806.

Although there is considerable scatter in the values of β for a given K, establishing a relationship between these variables is valuable for interpreting the discriminant of the cubic polynomial and assessing the applicability of Darcy's law.

5 Physical significance of the Discriminant

The physical significance of the quantity Δ is not apparent from Eqs. 13 - 15, but can be elucidated by casting β as a function of K and preparing contour plots of Δ while holding constant either the hydraulic conductivity, rainfall rate, or slope (Fig. 7). This exercise shows that positive values of Δ occur with smaller hydraulic conductivities, flatter slopes, and heavier rainfall. In contrast, negative values of Δ occur with larger hydraulic conductivities, steeper slopes and lower rainfall. Therefore, Δ is a measure of the capacity of the porous layer—influenced by the slope, hydraulic conductivity, and Forchheimer coefficient—in comparison to the rainfall.

6 The Discharge Ratio – Region of Applicability for Darcy's law

Any analysis of drainage through a porous layer will require a decision between Darcy's law or Forchheimer's equation to characterize the flow. The introduction of this article suggested a porous media Reynold's number to distinguish between the flow regimes. One challenge of this approach is selecting a length scale for using in Re_p. This section presents an alternative approach that avoids choosing a length scale.

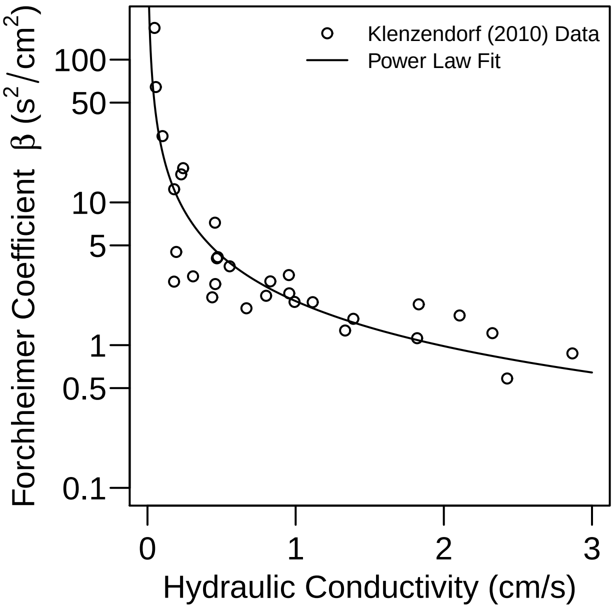

Figure 6: Comparison of Forchheimer coefficient and hydraulic conductivity for permeable friction course in the results of Klenzendorf (2010).

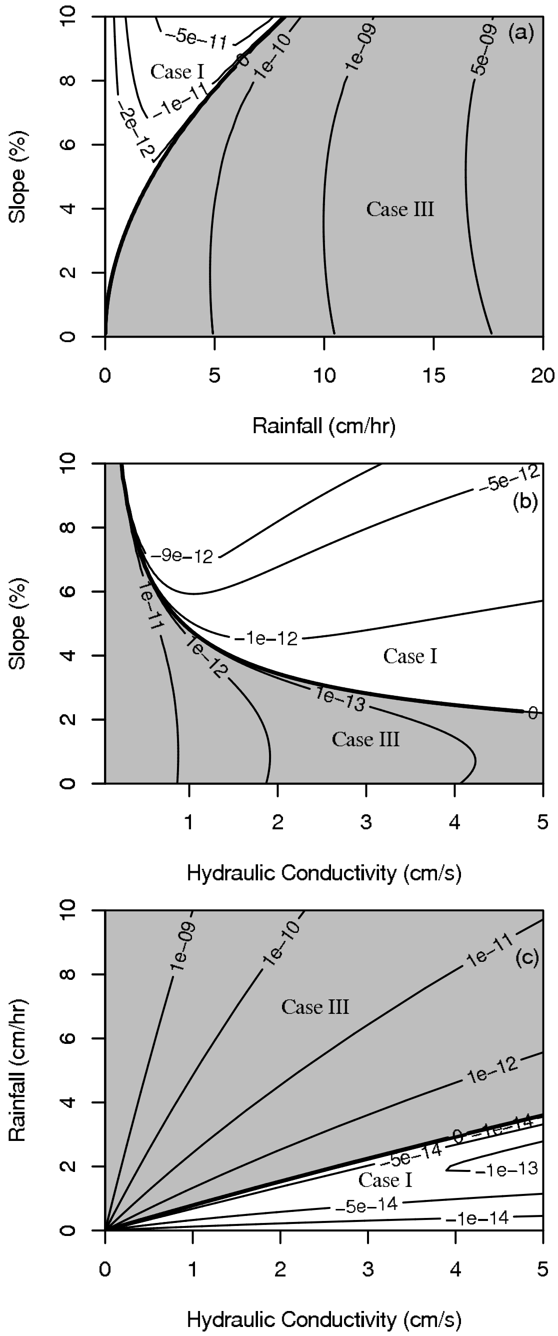

Figure 7: Contours of the discriminant (Eq. 13) for (a) hydraulic conductivity held constant at 1 cm/s; (b) rainfall held constant at 2cm/hr; and, (c) slope held constant at -3%. The discriminant is positive in areas shaded gray (Case III) and negative in white areas (Case I). The hydraulic conductivity and Forchheimer coefficient are related using Eq. 42.

The consequence of the inertial effects is to increase the resistance to flow, thereby reducing the specific discharge for the same hydraulic gradient. This result suggests a comparison of the specific discharge obtained by Forchheimer's equation with that obtained by Darcy's law. To facilitate the comparison, the Forchheimer specific discharge q_F is obtained from Eq. 3 using the quadratic formula. The positive radical is taken since a negative discharge is not meaningful in this case.

$$q_F = \frac{-\alpha + \sqrt{\alpha^2 + 4\beta I}}{2\beta} = \frac{\alpha}{2\beta}\left[\sqrt{1 + \frac{4\beta I}{\alpha^2}} - 1\right] \tag{43}$$

Since Darcy's law is $q = I/\alpha$, the specific discharge predicted by the two equations can be compared using a ratio, termed the Discharge Ratio (Φ).

$$\Phi = \frac{q_F}{q_D} = \frac{\alpha^2}{2\beta I}\left[\sqrt{1 + \frac{4\beta I}{\alpha^2}} - 1\right] \tag{44}$$

Since the Darcy specific discharge gives the limiting value of the specific discharge under Forchheimer's equation, the value of Φ ranges from 0 to 1. A value of $\Phi = 1$ corresponds to perfect agreement with Darcy's law while lower values of Φ imply inertial or turbulent flow effects. Accordingly, the behavior of Φ may be used to explore the applicability of Darcy's law.

The value of Φ depends upon the hydraulic gradient I and the coefficients $1/K$ and β. Invoking a correlation between the Forchheimer coefficient and the hydraulic conductivity reduces the discharge ratio to a function of two variables. In this case Eq. 42 has been used so Φ becomes specific to PFC layers.

$$\Phi_{PFC} = \frac{1/K^2}{2e^{0.710091}K^{-1.04806}I}\left[\sqrt{1 + \frac{4e^{0.710091}K^{-1.04806}I}{1/K^2}} - 1\right] \tag{45}$$

A surface plot of Eq. 45 is shown in Fig. 8. In the surface plot a value of $\Phi = 0.9$ corresponds to a 10% non-Darcy effect, which *Zeng and Grigg* (2006) suggest as a threshold. In the end, the acceptability of assuming a linear flow regime depends on the application. The non-Darcy effect can be quantified using the discharge ratio and an evaluation can be made.

This plot can be used to decide whether to use Darcy's law or Forchheimer's equation for a particular application. Similar plots can be prepared for other media where β and K have a different relationship.

7 Conclusions

This paper has examined the problem of Forchheimer seepage through an inclined porous layer under constant areal recharge. Forchheimer's equation has been used as model for inertial and turbulent flow effects and analytical solutions have been obtained for the steady state case. The analytical solutions

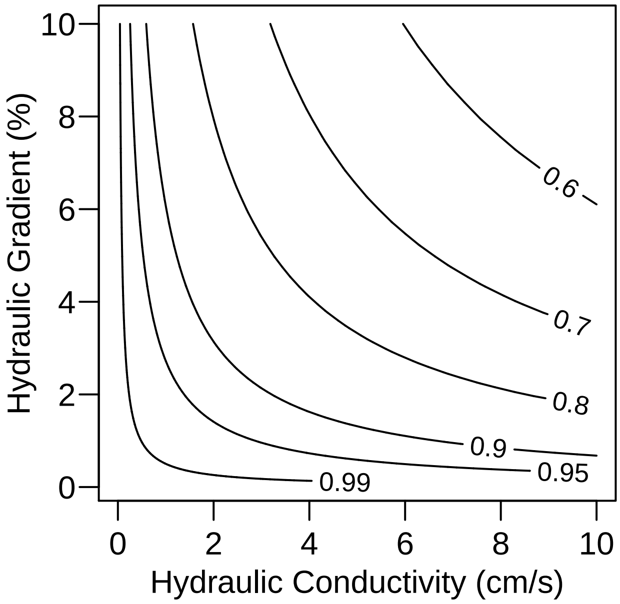

Figure 8: Contours of the discharge ratio ($\Phi = q_{Forchheimer}/q_{Darcy}$) as a function of hydraulic conductivity and hydraulic gradient for porous asphalt overlays. The Forchheimer coefficient and hydraulic conductivity have been related by Eq. 42.

match numerical results obtained by the Runge-Kutta method and agree with Darcy solutions when the Forchheimer coefficient is set to zero.

Forchheimer's equation adds a parameter, the Forchheimer coefficient, to the drainage problem. An estimate for this parameter in porous asphalt overlays was provided based on recent experimental results. The discriminant of the cubic polynomial appearing in the governing equation has been interpreted as a measure of the drainage capacity of the porous layer relative to the rainfall. Finally, a practical method for estimating the validity of Darcy's law has been proposed. Rather than computing a porous media Reynolds number which depends on the selection of length scale, the specific discharges obtained through Forchheimer's equation and Darcy's law are compared directly.

This work has been presented in the context of porous asphalt overlays, but the methods and results apply to other porous media fields such as hillslope hydrology and groundwater resources. Each application requires information on the hydraulic conductivity and Forchheimer coefficient. A parametric relationship between these values is needed for surface plots of the discriminant function and discharge ratio. The discharge ratio as a means of assessing the validity of Darcy's law could also be used for confined systems.

Notation

A,B,C,D,E,F,G	constants.
d	mean grain diameter [L].
h	saturated thickness [L].
I	hydraulic gradient .
K	hydraulic conductivity [L/T].
M	constant used to compute roots.
n	exponent in Izabash's law.
P, Q	constants used to compute roots.
r	rainfall rate [L/T].
R_1	dimensionless parameter.
R_2	dimensionless parameter.
S	slope [L/L].
U	flow rate per unit width [L^2 / T].
x	horizontal coordinate.
z	vertical coordinate.
Re_p	Reynold's number for porous media.
q	specific discharge [L/T].
α	linear coefficient in Forchheimer equation [T/L].
β	Forchheimer coefficient [T ^{2} / L ^{2}].
Δ	discriminant for cubic polynomials.
η	transformed variable.
$\lambda_1, \lambda_2, \lambda_3$	real roots of a cubic polynomial.
ν	kinematic viscosity [L^2/T].
Φ	discharge ratio.
φ	constant used to compute roots.

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