Polynomial Optimization for Water Networks: Global solutions for the valve setting problem

Bissan Ghaddar* Mathieu Claeys† Martin Mevissen‡ Bradley J. Eck §

Please cite this as: Bissan Ghaddar, Mathieu Claeys, Martin Mevissen, Bradley J. Eck, Polynomial optimization for water networks: Global solutions for the valve setting problem, In European Journal of Operational Research, Volume 261, Issue 2, 2017, Pages 450-459, ISSN 0377-2217, https://doi.org/10.1016/j.ejor.2017.03.011.

Abstract

This paper explores polynomial optimization techniques for two formulations of the energy conservation constraint for the valve setting problem in water networks. The sparse hierarchy of semidefinite programming relaxations is used to derive globally optimal bounds for an existing cubic and a new quadratic problem formulation. Both formulations use an approximation for friction loss that has an accuracy consistent with the experimental error of the classical equations. Solutions using the proposed approach are reported on four water networks ranging in size from 4 to 2000 nodes and are compared against a local solver, Ipopt and a global solver, Couenne. Computational results found global solutions using both formulations with the quadratic formulation having better time efficiency due to the reduced degree of the polynomial optimization problem and the sparsity of the constraint

*Corresponding Author. University of Waterloo, 200 University Avenue West, Waterloo, Ontario, N6L 3G1, Canada. E-mail: bghaddar@uwaterloo.ca

†Cambridge University, Trumpington Street, Cambridge, CB2 1PZ. E-mail: mathieu.claeys@gmail.com

‡IBM Research Ireland, IBM Technology Campus Building 3, Mulhuddart, Dublin 15, Ireland. E-mail: martmevi@ie.ibm.com

§IBM Research Ireland, IBM Technology Campus Building 3, Mulhuddart, Dublin 15, Ireland. E-mail: bradley.eck@ie.ibm.com

matrix. The approaches presented in this paper may also allow global solutions to other water network steady-state optimization problems formulated with continuous variables.

keywords: global optimization, polynomial optimization, semidefinite programming, valve setting problem, water networks.

1 Introduction

As water systems age and resources to make repairs diminish, mathematical optimization techniques are increasingly applied to design and operational problems for water networks. Design problems include placement of pressure reducing valves to minimize pressure or leakage [19] and selection of pipes and pipe sizes for minimum cost rehabilitation [6]. Operational problems include finding set points for pressure reducing valves, and scheduling pumps to minimize electricity costs [9] [13].

In the optimization of water networks the governing equations of energy and mass conservation act as constraints. Without considering any additional constraints, this set of equations has a unique solution provided frictional effects exist in the system [5]. Water network problems are formulated as non-linear and non-convex optimization problems. Hybrid simulation-optimization approaches have been developed to solve to water network problems by combining optimization with a simulator [17]. Pure mathematical optimization approaches are also becoming more popular owing to the improvement of optimization solvers and interest in the problems. The survey of [7] presents various mathematical programming approaches to problems in the field of water network optimization. Most of the proposed methods find local solutions for these types of problems because of the non-linear and non-convex problem structure. However, global solutions or bounds remain of theoretical and practical interest.

A particular challenge to finding global solutions to optimization problems on water networks is modeling the energy lost due to friction along the pipes. In fluid dynamics it is often convenient to describe energy in terms of the fluid weight. This quantity is called the *head* and has units of length. Head lost due to friction with the pipe is given by the Darcy-Weisbach law:

$$h_f = f\frac{l}{\phi}\frac{u^2}{2g} \tag{1}$$

where l is the pipe length, ϕ is the pipe diameter, u is the average velocity, and g is the gravitational acceleration all in consistent units. The friction factor, f has different forms for the laminar, transitional, and turbulent flow regimes. In most practical systems the turbulent regime dominates and the friction factor is found from the Colebrook formula [26]:

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{k_s}{3.7\phi} + \frac{2.51}{Re\sqrt{f}}\right). \tag{2}$$

Here k_s is the height of roughness elements inside the pipe, $Re = u\phi/\nu$ is Reynold's number and ν is the kinematic viscosity. Owing to the non-linear and implicit nature of Equation (2) mathematical optimization approaches introduce some form of approximation, such as the Hazen-Williams formula [4]

$$h_f = 10.65C^{-1.852}\phi^{-4.871}lQ^{1.85}. \tag{3}$$

The parameter C relates to the pipe roughness and $Q = u\frac{\pi\phi^2}{4}$ is the volumetric flow rate.

In this paper, we approximate the head loss formulas of Darcy-Weisbach and Hazen-Williams using the quadratic expression developed recently in [9]. The head loss equation is given as $h_f = aQ^2 + bQ + c$. For each pipe, values of a, b, and c are selected to minimize the relative error between Equation (4) and Equations (1) or (3) over the approximation range (the

minimum and maximum flows). The head loss equation is given as:

$$h_f = aQ^2 + bQ + c. \tag{4}$$

For each pipe, values of a and b are selected to minimize the relative error over the approximation range (the minimum and maximum flows). The form of Equation (4) balances the limiting case of fully rough flow ($Re \to \infty$), where a squared function applies, with improved accuracy obtained for more typical values of Re by including the linear term. For flow ranges encountered on water networks, the approximation error is typically less than 10%. Further details on the development and accuracy of the approximation are available in [9]. As discussed subsequently, applying Equation (4) results in non-linear and non-convex problems that may be tackled using polynomial optimization techniques.

We utilize the quadratic approximation to formulate the valve setting problem, the operational problem of choosing the set-point for pressure reducing valves, as a polynomial optimization problem. The physical motivation for studying valve control is to reduce leakage. Leakage may occur on distribution mains, service connections, or at the point of use. Although leakage localization and pipe replacement would be ideal, this is expensive and slow. Lowering system pressure by inserting control valves can reduce, though not eliminate, leakage before the pipes are replaced.

The valve setting problem has been studied extensively using mathematical programming starting with [22], who used a Taylor-series approximation of the head loss curve to apply sequential linear programming. A sequential linear programming technique was also used by [12] and by [14]. In [24], the authors introduced sequential quadratic programming for the valve setting problem. A parallel computing technique using sequential quadratic programming was given by [3], where different time steps were assigned to each node. Metaheuristic approaches have also been applied to the valve

setting problem. In [20], a genetic algorithm was used to find settings of isolating valves to minimize pressure heads. Additionally, [19] finds optimal locations and settings for control valves. A multi-objective approach considering the number, location, and setting of valves was proposed by [18]. A scatter-search algorithm was used by [16]. A cubic formulation of the valve setting problem was introduced in [8]. The resulting formulation was solved using both a local approach, Ipopt, and a global method by using polynomial optimization. The approach was tested on the Pescara water network.

In this paper, we study global solutions for a formulation of the valve setting problem that uses quadratic pipe friction and is solved by using semidefinite relaxations. These global solutions are shown to be close to the ones obtained by a simulation software where the accuracy of the quadratic approximation was assessed by comparing the flows and pressures from the optimization model with the EPANET simulator. The main contributions of the paper are as follows:

- showing that global solutions may be found for two formulations of the valve setting problem by using quadratic pipe friction and semidefinite relaxations;
- formulating energy conservation as a quadratic rather than cubic constraints;
- exploiting the structure of the network to take advantage of sparsity of the polynomial program;
- demonstrating that the proposed method outperforms available solvers (e.g., Ipopt and Couenne) and the resulting solutions are close to the ones obtained by the EPANET simulator.

The remainder of this paper is organized as follows. The cubic formulation proposed in [8] and a novel quadratic formulation of the valve setting

problem are presented in Section 2. The dense and the sparse semidefinite-based hierarchies are presented in Section 3. Computational results comparing the proposed approach with Ipopt, a local solver, and Couenne, a global solver, are presented for four benchmark networks in Section 4. Finally, a brief conclusion and future research directions are discussed in Section 5.

2 Mathematical Formulation

In this section we examine the valve setting problem in water distribution networks to find settings for pressure reducing valves that minimize the total pressure on the network while also providing a minimum pressure. The physical motivation for minimizing pressure is to reduce pressure driven leakage and to decrease the frequency of pipe bursts.

The water network, comprised of a set of nodes N and a set of pipes E, is modeled as a graph with $|N|$ vertices and $|E|$ edges. Nodes are numbered $i = 1, ..., |N|$ and nodal quantities include demand d_i (m^3/s), elevation e_i (m), and pressure head p_i (m). Edges are identified by source and target node (nodes i and j respectively) and edge quantities include flow rate $Q_{i,j}$, head loss as a function of $Q_{i,j}$ i.e., $h_f(Q_{i,j})$, and a valve indicator $v_{i,j}$. Edges that have a valve belong to set $V \subseteq E$ and the valve indicator on that edge is equal to 1. Since the location of the pressure reducing valves is fixed, the decision is the optimal setting of these valves, where the setting of each valve is deduced from the pressure at the downstream node.

Two formulations for valve setting that differ in treatment of flow direction within the energy conservation constraints are considered. In both formulations problem data include nodal demand and elevation, upper and lower bounds on flow and pressure, and constants of the approximation for energy loss. Variables are the pressure at each node and flow in each pipe. First, a cubic formulation [VS-C] given by [8] uses a quadratic approximation for head loss as in Equation (4) with the pair of inequality constraints

suggested by [21]. Second, a new quadratic formulation [VS-Q] is developed applying absolute values of the flow to Equation (4). Given

$$N : \text{is the set of nodes}$$
$$R : \text{is the set of reservoirs}$$
$$E : \text{is the set of pipes}$$
$$V : \text{is the set of valves,}$$

a valve setting optimization model written as a degree three polynomial program models the network as a directed graph [8]:

$$\textbf{[VS-C]} \ \min \sum_{i \in N} p_i \tag{5a}$$

$$\text{s.t.} \ p_{\min} \leq p_i \leq p_{\max} \qquad \forall i \in N \backslash R, \tag{5b}$$

$$p_i = 0 \qquad \forall i \in R, \tag{5c}$$

$$0 \leq Q_{i,j} \leq Q_{\max,i,j} \qquad \forall\, (i,j) \in E, \tag{5d}$$

$$\sum_k Q_{k,i} - \sum_l Q_{i,l} = d_i \qquad \forall\, i \in N, \tag{5e}$$

$$Q_{i,j}(p_i + e_i - p_j - e_j - h_f(Q_{i,j})) \geq 0 \qquad \forall\, (i,j) \in E, \tag{5f}$$

$$p_i + e_i - p_j - e_j - h_f(Q_{i,j}) \leq 0 \qquad \forall\, (i,j) \in E \backslash V. \tag{5g}$$

The sum of pressure heads (Equation (5a)) is minimized subject to a minimum and maximum pressure value imposed at each node as a service requirement (Equation (5b)). In our experiments, $p_{\min}$ and $p_{\max}$ are set to 15m (1.46 bar) and 100m (9.79 bar) respectively. Note that a pressure head value of 15m refers to the pressure created by the weight of a column of water having this height. Pressure head values may thus be converted to other units. Reservoirs are open to the atmosphere and so have zero pressure (Equation (5c)). The flow rate through each pipe is also bounded (Equation (5d)) and non-negative so that flows are positive in each link. Mass conservation

equates the net flow to each node with the imposed demand (Equation (5e)). Energy conservation is applied as a pair of inequality constraints (Equations (5f) and (5g)). Applying the head loss model of Equation (4), Equation (5f) is a cubic inequality constraint and Equation (5g) is a quadratic inequality constraint. Subtracting Equations (5f) and (5g) yields Bernoulli's equation where velocity heads are neglected according to usual practice for water networks. As shown in [21], a solution where both $Q_{i,j} > 0$ and $Q_{j,i} > 0$ is infeasible under the constraints (5f)-(5g). [VS-C] is a polynomial optimization problem of degree three and dimension $|N| + 2|E|$. Using the pair of inequality constraints (Equations (5f) and (5g)), the formulation [VS-C] avoids constraints that are not smooth at 0.

To reduce the problem degree and improve sparsity characteristics, we propose a new quadratic formulation for the valve setting problem based on an undirected graph. That is, for each pipe we only define one edge where the flow can be positive or negative instead of two edges with non-negative flow. Unlike the [VS-C] formulation, where two non-negative variables are needed to model each flow from (i, j) and (j, i), only one unrestricted variable is needed in the quadratic formulation. For each edge (i, j), there is a corresponding flow $Q_{i,j}$ that is positive if the flow is going from i to j, and negative otherwise. Additionally, in order to model the energy conservation, we use the absolute value of Q in the head loss equation which allows the

energy conservation to be reformulated as a quadratic constraint.

$$
\begin{aligned}
\textbf{[VS-Q]}\ \min & \sum_{i \in N} p_i & & & \text{(6a)}\\
\text{s.t. } & p_{\min} \leq p_i \leq p_{\max} & & \forall i \in N \backslash R, & \text{(6b)}\\
& p_i = 0 & & \forall i \in R, & \text{(6c)}\\
& -Q_{\max,i,j} \leq Q_{i,j} \leq Q_{\max,i,j} & & \forall\, (i,j) \in E, & \text{(6d)}\\
& \sum_k Q_{k,i} - \sum_l Q_{i,l} = d_i & & \forall\, i \in N, & \text{(6e)}\\
& p_j + e_j - p_i - e_i + h_f(Q_{i,j}) = 0 & & \forall\, (i,j) \in E \setminus V, & \text{(6f)}\\
& v_{i,j}(p_j + e_j - p_i - e_i) + h_f(Q_{i,j}) \leq 0 & & \forall\, (i,j) \in V, & \text{(6g)}\\
& v_{i,j} Q_{i,j} \geq 0 & & \forall\, (i,j) \in E, & \text{(6h)}
\end{aligned}
$$

where $h_f(Q_{i,j}) = \tilde{a}_{i,j} Q_{i,j} |Q_{i,j}| + \tilde{b}_{i,j} Q_{i,j} + \tilde{c}_{i,j}$ and the valve placement indicator $v_{i,j} \in \{-1, 0, 1\}$ is a parameter given for all $(i,j) \in E$. The valve placement indicator has a value of 1 if a valve is placed on (i,j), -1 if the valve is placed on (j,i), and 0 otherwise. Note that the problem is in polynomial form if absolute values are modeled as independent variables $\bar{Q}$ while imposing the additional constraints

$$
\begin{aligned}
\bar{Q}_{i,j}^2 &= Q_{i,j}^2 \\
\bar{Q}_{i,j} &\geq 0.
\end{aligned}
$$

Hence $h_f(Q_{i,j}, \bar{Q}_{i,j}) = \tilde{a}_{i,j} Q_{i,j} \bar{Q}_{i,j} + \tilde{b}_{i,j} Q_{i,j} + \tilde{c}_{i,j}$. As a result, [VS-Q] is a polynomial optimization problem with $|N| + 2|E|$ variables and degree two. It is easy to show that formulations [VS-C] and [VS-Q] are precisely equivalent.

In practice, networks having thousands of nodes are of interest. At that scale, both [VS-C] and [VS-Q] are highly challenging for global solvers of non-convex optimization problems such as Couenne [1] as they are not able

to find solutions even for small instances. Additionally, since both optimization problems are non-convex, local solvers such as Ipopt [2] are not guaranteed to find globally optimal solutions. Hence, we propose to solve [VS-C] and [VS-Q] using polynomial optimization techniques. Computationally, one can use the hierarchy of semidefinite (SDP) relaxations of Lasserre [15] to convexify the polynomial optimization problems [VS-C] and [VS-Q]. Unfortunately, the dimensions of these SDP relaxations grow rapidly with the size of the water network, posing a major computational challenge.

3 SDP Hierarchy and Exploiting Sparsity

Both valve setting formulations described in the previous section are polynomial programs. In this section, we present the dense and the sparse SDP hierarchies that are utilized to solve these two formulations. Consider the general polynomial programming problem whose objective and constraints are multivariate polynomials:

$$\begin{aligned} \textbf{[PP]} \ \min_{x} \quad & f(x) \\ \text{s.t.} \ \ & g_j(x) \geq 0, \qquad k = 1, \ldots, m. \end{aligned}$$

Let $S = \{x \in \mathbb{R}^n : g_j(x) \geq 0, j = 1, \ldots, m\}$ be the feasible set of [PP]. Then [PP] can be rephrased as

$$\begin{aligned} & \max_{\lambda} \lambda \\ & \text{s.t.} \ f(x) - \lambda \geq 0 \quad \forall\, x \in S. \end{aligned}$$

The condition $f(x) - \lambda$ being non-negative for all $x \in S$ is $\mathcal{NP}$-hard for most (interesting) choices of S. Lasserre [15] introduced a hierarchy of semidefinite relaxations corresponding to liftings of the polynomial programs into higher dimensions. The construction is motivated by results related to rep-

resentations of non-negative polynomials as sum-of-squares and the dual theory of moments. An advantage of approximating the non-negative polynomials using sum-of-squares (SoS) is that checking if a polynomial is sum-of-squares is equivalent to solving an SDP problem and hence it can be solved efficiently in polynomial time. Lasserre builds up a sequence of semidefinite relaxations of increasing size i.e., a sequence of convex optimization problems, of increasing size. Under mild conditions, which are slightly stronger than requiring S to be compact, the optimal values of these problems converge to the global optimal value of the original problem. Hence, using Lasserre's approach for general polynomial programs one may approach the global optimal value as closely as desired by solving a sequence of semidefinite problems that grow in the size of the semidefinite matrices and in the number of constraints. The larger the order of the relaxation, the better the lower bound. Let d be the degree of the polynomial optimization problem, then we obtain a SDP relaxation of order $r \geq r_0 = \lceil \frac{d}{2} \rceil$ as follows:

$$
\begin{aligned}
[\mathbf{L}_r] \quad & \max_{\lambda, \sigma_i(x)} \ \lambda \\
& \text{s.t. } f(x) - \lambda = \sigma_0(x) + \sum_{j=1}^{m} g_j(x)\sigma_j(x) \\
& \sigma_0(x) \in SoS_{2r},\ \sigma_j(x) \in SoS_{2r-\deg(g_j)},
\end{aligned}
$$

where $r_0 = \lceil \frac{d}{2} \rceil$ is called the minimal relaxation order of the polynomial program.

At each level of the Lasserre's hierarchy, the SDP relaxation increases in size, making the relaxation computationally expensive to solve. Thus, the current scalability of current SDP solvers limits the tractability of the Lasserre hierarchy even for low values of r and medium-scale polynomial programs. However, this size can be reduced substantially by exploiting sparsity in the underlying problem, i.e., having a small number of variables

in each constraint only, as proposed by Waki et al. [25]. The resulting hierarchy is referred to the sparse hierarchy. Let $\{I_k\}_{k=1}^p$ be the set of maximal cliques of the correlative sparsity pattern graph of [PP] following the construction in [25], i.e. $I_k \subset \{1, \ldots, n\}$. The sparse hierarchy of SDP relaxations is then given by

$$
\begin{aligned}
[\mathbf{S}_r] \quad & \max_{\lambda, \sigma_k(x), \sigma_{j,k}(x)} \lambda \\
& \text{s.t. } f(x) - \lambda = \sum_{k=1}^{p} \left(\sigma_k(x) + \sum_{j \in J_k} g_j(x) \sigma_{j,k}(x) \right) \\
& \sigma_k \in SoS_{2r}(I_k), \sigma_{j,k} \in SoS_{2r-\deg(g_j)}(I_k).
\end{aligned}
$$

where $SoS_{2r}(I_k)$ is the set of all sum-of-squares polynomials of degree up to $2r$ supported on I_k and $(J_1, \ldots, J_p)$ is a partitioning of the set of polynomials $\{g_j\}_j$ defining S such that for every j in J_k, the corresponding g_j is supported on I_k. While [S$_r$] provides a weaker relaxation to [PP] than [L$_r$] for a fixed relaxation order r in general, the asymptotic convergence result for the dense hierarchy extends to the sparse case, that is, as the relaxation order increases the bound converges to the optimal solution of [PP] under mild conditions [25].

Many real-world water networks are represented by sparse graphs because the degree of most nodes in the networks is small. The sparsity of a water network results in [VS-C] and [VS-Q] being sparse polynomial optimization problems and hence the sparse hierarchies are promising to derive global optimal solutions. Therefore, the sparse hierarchy of SDP relaxations of Waki et al. [25] is used, which improves the tractability of the Lasserre's hierarchy by exploiting sparsity of the problem.

Even though both formulations exhibit sparsity and have the same number of variables, $|N| + 2|E|$, formulations [VS-Q] and [VS-C] have different degrees and sparsity patterns. For example, Figure 1 shows the chordally

extended correlative sparsity pattern of [VS-C] vs [VS-Q] for a water network with 71 nodes and 99 edges. The figure confirms that [VS-Q] exhibits a sparser structure (8805 non-zero elements compared to 15275 non-zero elements). This difference in sparsity arises from the network structure implied by mass conservation at each node. Since mass conservation is encoded with twice as many variables in [VS-C] than [VS-Q] (two signed flows versus a single unsigned flow assigned to each pipe), [VS-Q] is expected to outperform [VS-C] for sparse SDP relaxations.

Another potential advantage of [VS-Q] with respect to [VS-C] in the context of hierarchy of SDP relaxations is that the degree of the problem is reduced from three to two. Thus, the first-order relaxation in the Lasserre hierarchy of relaxations can be used to find a lower bound for [VS-Q] while the second-order relaxation is needed to obtain a lower bound for [VS-C] [15]. Therefore, the minimal relaxation order is one less for [VS-Q] with respect to [VS-C], which allows to solve an extra relaxation order. Increasing the relaxation order in this hierarchy enables global solution of the valve setting problem. However, as seen in our numerical examples in Section 4, we did not find a problem instance which was solved to optimality by [VS-Q] at an earlier relaxation order than [VS-C], that is, the first-order relaxation of [VS-Q] provided lower bounds while the second-order relaxation of [VS-Q] and [VS-C] resulted in global optimal solutions. Additionally, using the second-order of the hierarchy, the SDP relaxation of [VS-Q] has a smaller constraint matrix compared to the second-order hierarchy of [VS-C] which significantly reduces the computational time. Using the same water network as Figure 1, this is illustrated in Figure 2 where the constraint matrix of [VS-Q] is of dimension 29621×266398 compared to 145146×1260423 for [VS-C] for the second-order of the hierarchy.

In the next section, both models are implemented and solved for four water networks. Computational results of both cubic and quadratic models

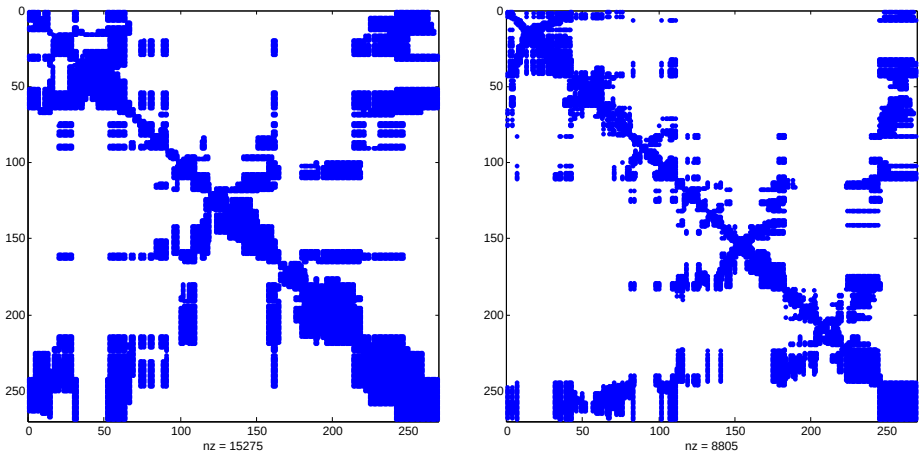

Figure 1: Chordally extended correlative sparsity patterns of [VS-C] (left) and [VS-Q] (right).

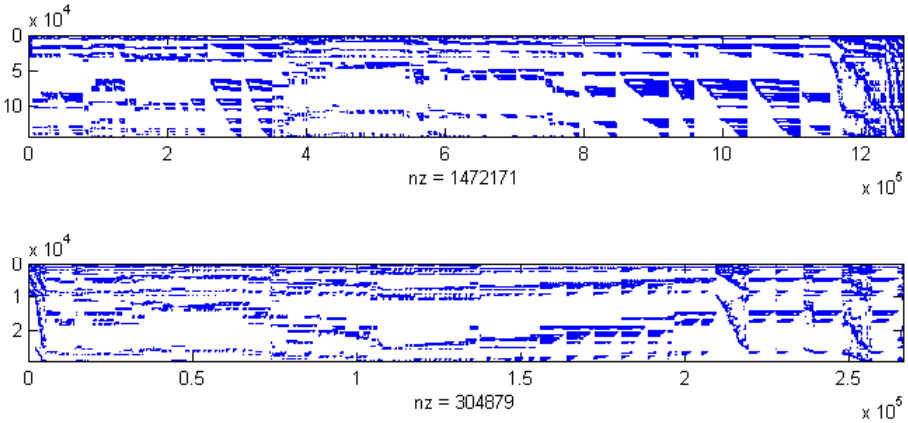

Figure 2: Sparsity pattern of the constraint matrix of the second order relaxation of [VS-C] (top) and [VS-Q] (bottom).

are conducted and compared with each other and with the local solver Ipopt and the global solver Couenne.

4 Computational Results

This section presents numerical experiments conducted on four water networks using the formulations [VS-C] and [VS-Q] given in Section 2. The first network is a small size water network from our industrial partners, the second network is a benchmark network with 25 nodes, the third network is a medium size network (Pescara water network), and the fourth network is a large scale network (Exnet) taken from the literature. The valve placement locations for each network (i.e., the edges where the valves are placed) are chosen based on the approach presented in [8].

Computations were performed running Red Hat Linux on a blade server with 100GB of RAM and a processor speed of 3.5GHz. SeDuMi [23] is used to solve the dense hierarchy of SDP relaxations, $[\mathrm{L}_r]$. SparsePOP [25] is used to construct and solve the sparse hierarchy, $[\mathrm{S}_r]$. SparsePOP accepts a polynomial program as input and builds a sparse SDP relaxation of the polynomial program. The resulting SDP relaxation can be solved using SeDuMi or SDPA [11]. In our case, SeDuMi is used for all networks and for Network 3, SDPA is also used and compared with SeDuMi. Global optimality (reported by (*)) for $[\mathrm{S}_r]$ is verified if SparsePOP output values of POP.absError and POP.scaledError less than 10^{-3} which indicates that the solution found by SparsePOP is feasible within a relative error of 10^{-3} and the objective value is equivalent to the lower bound given by the SDP relaxation. For the case of $[\mathrm{L}_r]$, the global optimal solution is verified by computing the feasibility error of the SDP solution obtained by SeDuMi. The time limit for all instances is set to 10 hours.

Additionally, a comparison of the solution of the optimization model [VS-Q] and of the simulation using EPANET is provided for the first three net-

works to show the accuracy of the quadratic approximation. Furthermore, the local solver Ipopt and the global solver Couenne are used to compare the obtained solution with the SDP relaxation solution.

4.1 Network 1

The first network (see Figure 3) consists of one reservoir, three junctions, and four pipes and is based on a real network provided by one of our industrial partners. The network size is small and in this instance we consider three cases with a placement of zero, one, or two valves. For each case, [VS-C] and [VS-Q] are solved using a hierarchy of SDP relaxations with and without exploiting sparsity [25, 15] ([S_r] and [L_r] respectively). The first and the second order dense and sparse hierarchies of the SDP relaxation are solved for [VS-Q] (L_1, L_2, S_1, and S_2 respectively) and the second-order dense and sparse hierarchies (L_2 and S_2 respectively) are solved for [VS-C]. Note that for [VS-C], the lowest order of the hierarchy that one can start with is L_2 or S_2. The optimal solution for the SDP hierarchies for [VS-C] and [VS-Q] is a lower bound on the original problem. In this case, [L_r] and [S_r] (for r equal one and two) provided the same lower bound, z_l. The lower bound is reported in addition to the computational time with and without exploiting sparsity (t_s and t respectively). Additionally, [VS-C] and [VS-Q] are solved using the local solver Ipopt and the upper bound, z_u, is reported. The computational time of Ipopt is not reported as it is few seconds even for the large instance (Network 4). The global solver, Couenne, is used to solve [VS-C] and [VS-Q] and in the case of zero optimality gap a global solution is reported.

Without introducing any valves, the sum of pressure heads is 193.8m. Introduction of a single valve causes a slight improvement in the objective but adding a second valve corresponds to a 60% improvement.

The computational results shown in Table 1 show that using SeDuMi

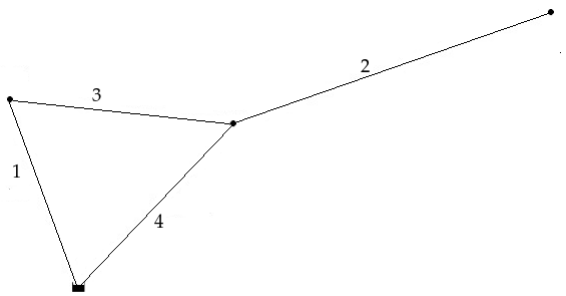

Figure 3: IND1 Water Network

Table 1: [VS-C] and [VS-Q] results using SeDuMi.

	[VS-C]					[VS-Q]							
	L_2/S_2			Ipopt	Couenne	L_1/S_1			L_2/S_2			Ipopt	Couenne
Valve	z_l	t	t_s	z_u	z_u	z_l	t	t_s	z_l	t	t_s	z_u	z_u
0	193.8*	5.6	4.5	193.8	193.8*	122.64	0.5	0.6	193.8*	47.1	2.1	193.8	193.8*
1	193.5*	27.7	8.2	193.5	193.5*	109.41	0.7	0.7	193.5*	57.8	2.0	193.5	193.5*
2	76.7*	9.6	6.3	76.7	76.7*	52.90	2.1	1.5	76.7*	94.2	2.9	-	76.7*

*: global optimal solution.
−: program terminated unsuccessfully.

with and without exploiting sparsity (i.e., with and without SparsePOP) provides optimal solutions for the two different valve placements for [VS-Q] and [VS-C] using L_2 and S_2. Using the same order of the hierarchy, L_2 or S_2, [VS-Q] is more efficient than [VS-C].

Additionally, Table 1 presents computational results obtained by Ipopt and Couenne. Using Couenne, the solutions are provably optimal in this case as the solver terminated successfully with a zero gap for all cases. The average computational time in this case was less than one second. For the case of Ipopt, the solutions are optimal with the exception for [VS-Q] with two valves where Ipopt terminates unsuccessfully. However, using Ipopt only, there is no guarantee of optimality, and without the results of the SDP relaxation, the Ipopt results can not be verified as being globally optimal. On the contrary, the SDP relaxation provides lower bounds and can also prove optimality by extracting a feasible solution with the same objective

value for the cases of [VS-C] and [VS-Q] using the L_2 and S_2 relaxation.

Table 2: Comparison of pressures and flows as obtained by [VS-Q] using S_2 and EPANET 2.0 simulator.

	Pressure			Flow	
Node	[VS-Q]	EPANET	Link	[VS-Q]	EPANET
1	26.83	24.83	1	89191.45	89332.33
2	0.00	0.00	2	-89332.34	-89332.34
3	15.00	15.00	3	89191.45	89332.33
4	34.86	34.86	4	0.00	0.00

Table 2 provides a comparison between the flow and pressure solution derived by the optimization model [VS-Q] using order two, S_2, and those obtained by EPANET for the two valve case. The results are very close (the maximum error is 8% for the pressure and is less than 1% for the flow) which verify that the solutions computed using the optimization model are consistent with the EPANET simulator.

4.2 Network 2

The second network is a benchmark network presented in Sterling and Bargiela [22], referred to as SB network. It consists of 25 junctions, 3 reservoirs, and 37 pipes. We solve [VS-C] and [VS-Q] to optimality using zero, one, two, and three valve placement. The valves are located at pipes 1, 5, and 11. For each case, we solve the SDP relaxation S_2 for [VS-C] and the SDP relaxation S_1 and S_2 for [VS-Q]. In this case we only report results using sparsity (i.e., using [S_r] and SparsePOP), as the dimension of the problem is larger than that of the previous network. Therefore, solving the problem without exploiting sparsity is computationally expensive and can not be done within the set time limit.

The sum of pressure heads in the network dropped from 857.8m to 431.7m when introducing three valves. Table 3 presents the results of the various relaxations in addition to Ipopt and Couenne results. Couenne ob-

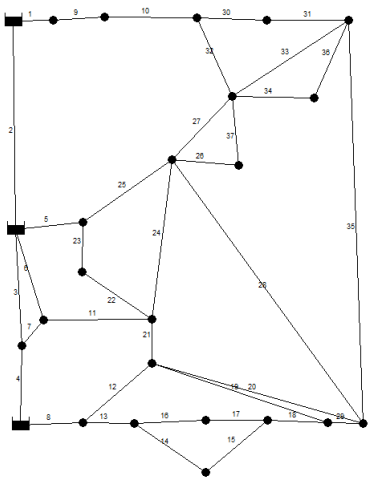

Figure 4: SB Water Network

Table 3: [VS-C] and [VS-Q] results using SeDuMi.

	[VS-C]				[VS-Q]					
	S_2		Ipopt	Couenne	S_1		S_2		Ipopt	Couenne
Valve	z_l	t_s	z_u	z_u, z_l	z_l	t_s	z_l	t_s	z_u	z_u, z_l
0	857.8*	5736.1	857.6	857.8,528.5	537.4	16.1	857.8*	66.3	857.6	857.6*
1	806.0*	5874.0	805.8	806.0,493.3	497.0	14.1	806.0*	70.9	805.8	805.8*
2	783.7*	6073.7	817.7	783.8,586.7	497.1	14.0	783.7*	78.4	783.7	783.6*
3	431.7*	6089.8	695.9	431.85,363.9	364.1	18.1	431.7*	79.1	432.0	431.8,428.2

*: global optimal solution.

tained lower and upper bounds using [VS-C] while it obtained the global optimal solution using [VS-Q] (except for the case of three valves). This result is expected as global solvers tend to perform better on quadratic formulations compared to other types of non-linear formulations. Hence, Couenne in this case benefited from the new proposed formulation. However, in terms of computational time Couenne was much slower than S_2 for [VS-Q] as it required more than 600 seconds on average whereas S_2 for [VS-Q] required 74 seconds on average. For the case of the local solver, Ipopt obtained better bounds for [VS-Q] than that of [VS-C] for placing 2 and 3 valves. Additionally, we note that the computational time taken by Ipopt to solve [VS-Q] is faster than solving [VS-C] (average time was 27 sec

versus 0.8 sec). The lower bound obtained from the SDP relaxation using SeDuMi at the second-order of the hierarchy is globally optimal. Comparing S_2 for [VS-C] versus S_2 for [VS-Q] using SeDuMi, the bound obtained by both relaxations is the same while the computational time is significantly higher for [VS-C], the average time is 5943 compared to 73 seconds. Note that for valve placements 0 and 1, Ipopt provides a slightly lower value than the SeDuMi solution however, this falls within the relative error of 10^{-3} of SparsePOP.

Figure 5 provides a comparison between the flow and pressure solution derived by the optimization model [VS-Q] using S_2 with those obtained by EPANET for the three valve case to assess the accuracy of the quadratic approximation. The maximum errors for the pressure and the flow data are less than 2% and the average is less than 0.5%. The results verify that the solutions computed using the optimization model are consistent with the EPANET simulator.

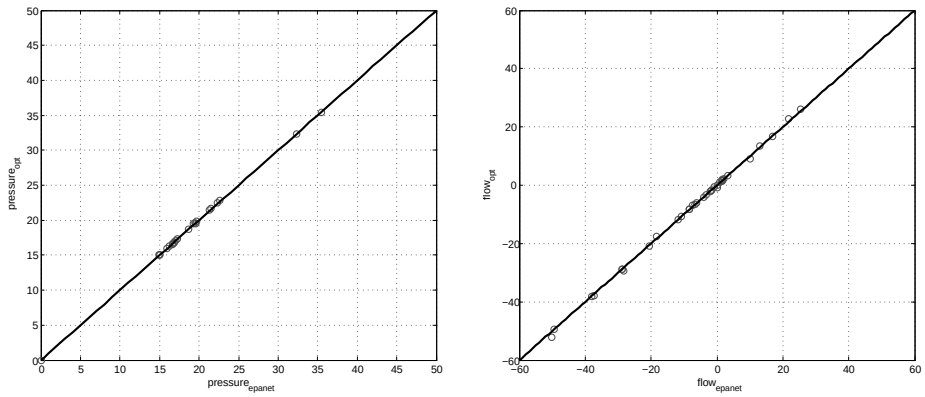

Figure 5: Comparison of pressures and flows for SB network as obtained by [VS-Q] using S_2 and EPANET 2.0 simulator.

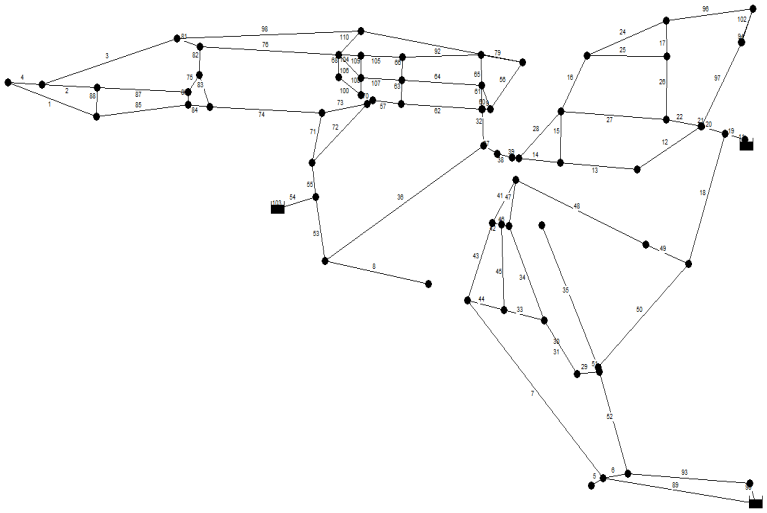

Figure 6: Pescara Water Network

4.3 Network 3

The third network is the Pescara network (see Figure 6), which is a reduced version of a network for a medium size Italian city. The network is first mentioned by Bragalli et al. [6] and consists of 68 junctions, 3 reservoirs, and 99 pipes. We solve [VS-C] and [VS-Q] to optimally using zero, one, and two valve placement. The valves are located at pipes 90 and 97. For each case, we solve S_2 for [VS-C] and S_1 and S_2 for [VS-Q].

Table 4: [VS-C] and [VS-Q] results using SeDuMi.

	[VS-C]				[VS-Q]					
	S_2		Ipopt	Couenne	S_1		S_2		Ipopt	Couenne
Valve	z_l	t_s	z_u	z_u	z_l	t_s	z_l	t_s	z_u	z_u
0	2012.9*	10048.1	2012.4	−−	1266.6	21.4	2012.9*	1225.7	2012.4	2013.0*
1	−−	−−	1800.6	−−	1230.2	49.1	1801.0*	1609.3	1800.6	−−
2	−−	−−	1663.1	−−	1230.2	30.8	1663.4*	1039.7	1663.1	−−

*: global optimal solution.
−−: time limit exceeded.

As expected, placing additional valves reduces the objective value. For instance, with the placement of two valves, a reduction of 12% in the objec-

Table 5: [VS-C] and [VS-Q] results using SDPA.

	[VS-C]				[VS-Q]					
	S_2		Ipopt	Couenne	S_1		S_2		Ipopt	Couenne
Valve	z_l	t_s	z_u	z_u	z_l	t_s	z_l	t_s	z_u	z_u
0	1954.8	9420.1	2012.4	−−	1266.6	1.6	2005.1	371.5	2012.4	2013.0*
1	1752.8	7006.4	1800.6	−−	1229.8	1.6	1794.9	403.1	1800.6	−−
2	1626.7	9493.1	1663.1	−−	1229.8	2.8	1656.8	383.9	1663.1	−−

*: global optimal solution.
−−: time limit exceeded.

tive is achieved.

Tables 4 presents the Ipopt and Couenne results and provides the lower bound given by the SDP relaxation for each level of the hierarchy using SeDuMi. The lower bound obtained from the SDP relaxation using SeDuMi at the second-order of the hierarchy is globally optimal. Using [VS-C], the time limit of ten hours is reached before reporting a solution, whereas [VS-Q] with S_2 found the optimal solution in 20 minutes. In order to solve [VS-C], we used the SDPA solver to provide solutions for the S_2 relaxation (see Table 5). SDPA is faster as a solver but at the expense of solution quality since the bounds obtained are lower than those obtained by SeDuMi. On the other hand, SeDuMi provides better bounds for these problems but is computationally slower than SDPA and hence reached the time limit for the S_2 relaxation of [VS-C] whereas SDPA solved these instances within 3 hours of computational time. Note that for all cases, Ipopt has a slightly lower value than SeDuMi the solution however, this falls within the relative error of 10^{-3} provided by SparsePOP. In this case, Couenne reached the time limit and is not able to solve the instances with one and two valves for [VS-Q] and all the instances for [VS-C].

A comparison between the flow and pressure solution is presented in Figure 7 where the solution obtained by the optimization model [VS-Q] using S_2 is compared with the values obtained by EPANET for the two valve case. The maximum error for the pressure data is 9.2% and for the

flow data is 6.7% and the average error is 1.4% for the pressure and 3.3% for the flow. Similar to the SB network, the results confirm that the solutions computed using the optimization model are consistent with the EPANET simulator.

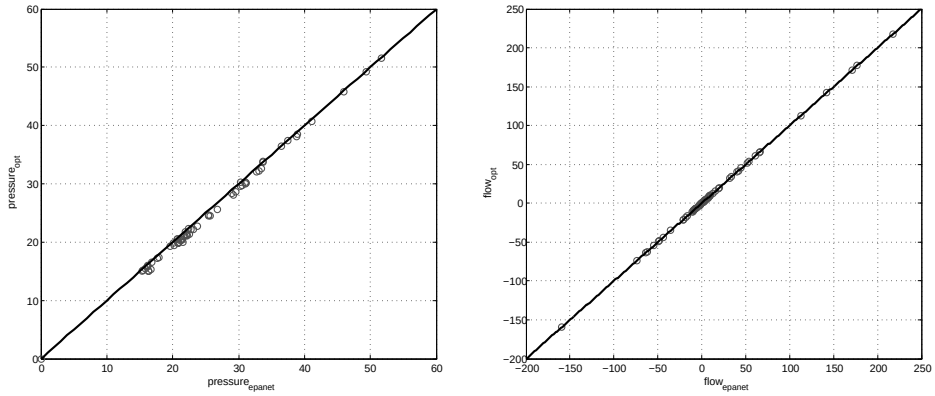

Figure 7: Comparison of pressures and flows for Pescara network as obtained by [VS-Q] using S_2 and EPANET 2.0 simulator.

4.4 Network 4

The fourth network is Exnet (see Figure 8) which is mentioned in Farmani et al. [10]. The network consists of 1893 nodes and 2466 pipes, and represents a realistic benchmark network [10]. In this case there are four valves on pipes 104, 226, 1717, and 2045. We only report results using sparsity for S_1 as the dimension of the problem makes it computationally expensive to run the rest of the relaxations.

Table 6: [VS-Q] results using SeDuMi.

	[VS-Q]			
	S_1		Ipopt	Couenne
Valve	z_l	t_s	z_u	z_u
0	30307.1	3189.2	71894.6	−−
4	29831.3	25528.9	76847.6	−−

−−: time limit exceeded.

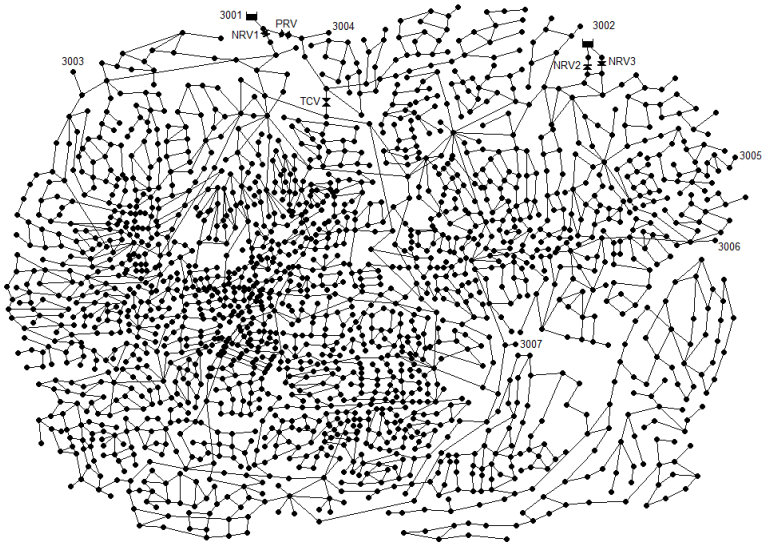

Figure 8: Exnet Water Network

For this case solving the first order relaxation of [VS-C] and the second order relaxation of [VS-Q] is not possible as we run out of memory and hence only S_1 can be solved using SeDuMi as shown in Table 6. S_1 provides a lower bound in this case and the Ipopt provides an upper bound, the gap between the lower and upper bound is 57% for zero valves and 61% for four valves. Couenne was not able to solve [VS-Q] or [VS-C] for this instance.

5 Conclusions

In this paper, we apply polynomial optimization techniques to the valve setting problem to reduce leakage on aging systems. For the explored problem, an existing cubic and new quadratic formulation of the energy conservation constraints are examined. Global solutions were found for both formulations by exploiting sparsity and using the hierarchy of SDP relaxations. In our experiments, the quadratic formulation had better performance with an optimality gap of zero for three out of the four benchmark instances us-

ing the sparse hierarchy of SDP relaxations. Furthermore, the global solver Couenne and the local solver Ipopt both benefited from the quadratic formulation in terms of the solution quality (better bounds and in some cases global optimal solutions) and the computational time. Additionally, the solution of the optimization model is compared with the EPANET simulator and the resulting maximum error was less than 10% for the pressure and the flow values while the average error was less than 5%.

The valve setting problem studied here had a linear objective, polynomial constraints, and continuous variables. In future work, we plan to explore the applicability of polynomial techniques for finding global solutions to related problems in water networks. Examples include a special case of pressure driven leakage and model calibration for nodal demands. Other extensions include design problems with binary decisions such as valve placement and pipe sizing.

Acknowledgment

We greatly thank the referees and the Editor for their thorough and thoughtful comments that helped us improve the quality of the paper.

References

[1] *Couenne (v. 0.4)*, 2011.

[2] *Ipopt (v. 3.10)*, 2011.

[3] J. M. Alonso, F. Alvarruiz, D. Guerrero, V. Hernández, P. A. Ruiz, A. M. Vidal, F. Martínez, J. Vercher, and B. Ulanicki. Parallel computing in water network analysis and leakage minimization. *Journal of Water Resources Planning and Management*, 126(4):251–260, 2000.

[4] A. W. W. Association et al. *Steel pipe; a guide for design and installation*, volume 11. AWWA, 1989.

[5] B. L. Berghout and G. Kuczera. Network linear programming as pipe network hydraulic analysis tool. *Journal of Hydraulic Engineering*, 123(6):549–559, 1997.

[6] C. Bragalli, C. D'Ambrosio, J. Lee, A. Lodi, and P. Toth. On the optimal design of water distribution networks: a practical MINLP approach. *Optimization and Engineering*, 13(2):219–246, 2012.

[7] C. D'Ambrosio, A. Lodi, S. Wiese, and C. Bragalli. Mathematical programming techniques in water network optimization. *European Journal of Operational Research*, 243(3):774 – 788, 2015.

[8] B. J. Eck and M. Mevissen. Fast non-linear optimization for design problems on water networks. In *World Environmental and Water Resources Congress*, 2013.

[9] B. J. Eck and M. Mevissen. Quadratic approximations for pipe friction. *Journal of Hydroinformatics*, 17(3):462–472, 2015.

[10] R. Farmani, D. Savic, and G. Walters. EXNET benchmark problem for multi-objective optimization of large water systems. In *Modelling and Control for Participatory Planning and Managing Water Systems, IFAC workshop, Italy*, 2004.

[11] K. Fujisawa, K. Nakata, M. Yamashita, and M. Fukuda. SDPA project: Solving large-scale semidefinite programs. *Journal of Operations Research Society of Japan*, (50):278–298, 2007.

[12] G. Germanopoulos and P. Jowitt. Leakage reduction by excess pressure minimization in a water supply network. *ICE Proceedings*, 87:195–214, June 1989.

[13] B. Ghaddar, J. Naoum-Sawaya, A. Kishimoto, N. Taheri, and B. Eck. A lagrangian decomposition approach for the pump scheduling problem in water networks. *European Journal of Operational Research*, 241(2):490 – 501, 2015.

[14] P. W. Jowitt and C. Xu. Optimal valve control in water-distribution networks. *Journal of Water Resources Planning and Management*, 116(4):455–472, 1990.

[15] J. Lasserre. Global optimization problems with polynomials and the problem of moments. *SIAM Journal on Optimization*, 11:796–817, 2001.

[16] S. Liberatore and G. Sechi. Location and calibration of valves in water distribution networks using a scatter-search meta-heuristic approach. *Water Resources Management*, 23:1479–1495, 2009.

[17] J. Naoum-Sawaya, B. Ghaddar, E. Arandia, and B. Eck. Simulation optimization approaches for water pump scheduling and pipe replacement problems. *European Journal of Operational Research*, 246(1):293 – 306, 2015.

[18] M. Nicolini and L. Zovatto. Optimal location and control of pressure reducing valves in water networks. *Journal of Water Resources Planning and Management*, 135(3):178–187, 2009.

[19] L. F. R. Reis, R. M. Porto, and F. H. Chaudhry. Optimal location of control valves in pipe networks by genetic algorithm. *Journal of Water Resources Planning and Management*, 123(6):317–326, 1997.

[20] D. A. Savic and G. A. Walters. An evolution program for optimal pressure regulation in water distribution networks. *Engineering Optimization*, 24(3):197–219, 1995.

[21] H. Sherali and E. Smith. A global optimization approach to a water distribution network design problem. *Journal of Global Optimization*, 11:107–132, 1997.

[22] M. Sterling and A. Bargiela. Leakage reduction by optimised control of valves in water networks. *Transactions of the Institute of Measurement and Control*, 6(6):293–298, 1984.

[23] J. Sturm. Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11(1-4), 1999.

[24] K. Vairavamoorthy and J. Lumbers. Leakage reduction in water distribution systems: Optimal valve control. *Journal of Hydraulic Engineering*, 124(11):1146–1154, 1998.

[25] H. Waki, S. Kim, M. Kojima, and M. Muramatsu. SparsePOP : a sparse semidefinite programming relaxation of polynomial optimization problems. *ACM Transactions on Mathematical Software*, 35(2):1–13, 2008.

[26] F. M. White. *Fluid Mechanics*. McGraw-Hill, USA, 4th edition, 1999.