Use of a smoothed model for pipe friction loss

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ABSTRACT

This note examines a globally smooth approximation for friction loss in pipelines suggested by Burgschweiger and co-workers. The approximation contains five parameters, two of which are open to selection. A method is provided for choosing these based on laminar conditions near zero flow. Using the proposed approach, approximation accuracy is reported in terms of relative roughness and Reynold's number.

INTRODUCTION

Modeling friction loss within derivative-based optimization models remains an area of active research. Head lost due to friction is described by the Darcy-Weisbach formula (White 1999)

$$h = f \frac{L}{\phi} \frac{V^2}{2g} \tag{1}$$

where f is the Darcy friction factor; L is the pipe length, ϕ is the diameter, V is the average velocity; and g is the gravitational constant, all in consistent units. The friction factor f varies according to the flow regime. Under laminar conditions $f = 64/Re$, where $Re = VD/\nu$ is the Reynolds number, and ν is the kinematic viscosity. In the turbulent regime, the friction factor's behavior with respect to Re varies according to flow conditions and roughness height k as described by the Colebrook-White formula:

$$\frac{1}{\sqrt{f}} = -2\log_{10}\left(\frac{k}{3.7\phi} + \frac{2.51}{Re\sqrt{f}}\right) \tag{2}$$

Challenges of applying Eq 2 within optimization models include the implicit nature of the relation, its computational complexity (c.f. (Giustolisi et al. 2011)), and the differentiability and smoothness of approximations. Coping strategies for optimization approaches include using the explicit Hazen-Williams formula and approximating the Darcy-Weisbach curve.

Several earlier workers have suggested approximations of the Darcy-Weisbach curve aimed at optimization problems. An explicit power law form is suggested by Valiantzas (2008). The limiting case of rough pipes as described by the Prandtl-Karman relation is applied as an approximation by Gleixner et al. (2012). Eck and Mevissen (2015) propose a quadratic approximation of Darcy-Weisbach friction losses, minimizing the relative error. Both the power law and quadratic forms treat flow as a non-negative variable. In these cases, flow direction may be modeled using absolute value (Vairavamoorthy and Lumbers 1998) or by restricting flows to positive values and including a network link for each direction (Ghaddar et al. 2015). The smoothed approximation of Burgschweiger et al. (2009) provides a different modelling approach that allows flows to be real-valued without introducing an absolute value function. Comparing to other approximations of the Darcy-Weisbach curve aimed at optimization problems, the approximation also has the advantages of smoothness in the second derivative and asymptotic correctness. Smoothness is desirable in optimization problems to improve convergence behavior and to enable a broader choice of algorithms. Burgschweiger et al. (2009) report improved convergence performance with the smoothed model, even for methods using only first derivatives. In their study of water network design Bragalli et al. (2011) require a smooth friction loss model in order to apply an interior point solver. The behavior of the smoothed approximation in the practical and asymptotic flow ranges is a topic of this paper. As discussed below, the approximation uses five parameters. Three of these are set by pipe properties while the remaining two are open to selection.

The remainder of this note examines the smoothed model suggested by Burgschweiger et al. To the author's knowledge a detailed procedure for choosing the open coefficients of the smoothed model does not exist elsewhere. Thus, a method based on laminar conditions near zero flow is developed. The approximation accuracy is also reported.

SMOOTHED MODEL FOR FRICTION LOSS

In their work to optimize the operation of the water distribution system in Berlin, Burgschweiger and co-workers developed the following approximation for Darcy-Weisbach friction loss in water pipes (Burgschweiger et al. 2009).

$$\hat{h}(Q) = r\left(\sqrt{Q^2+a^2} + b + \frac{c}{\sqrt{Q^2+d^2}}\right)Q \tag{3}$$

The approximate head loss $\hat{h}$ depends on the volumetric flow rate Q and the five parameters r, a, b, c, and d. The functional form of Eq 3 has several advantages for derivative based optimization models. It is globally smooth in the second derivative and handles positive and negative flows without an absolute value sign. Values of r, b, and c are chosen from pipe and fluid properties and asymptotic correctness while a and d are open to selection.

Considering a pipe of length, L, diameter ϕ, and roughness height k the resistance parameter, r is defined within $h = rQ|Q|$ using Prandtl-Karman's value of the Darcy friction factor for hydraulically rough pipes.

$$r = \left(2\log\frac{k}{3.71\phi}\right)^{-2}\frac{8L}{\pi^2\phi^5 g} \tag{4}$$

The parameters b and c are found such that the theoretical curve is approximated up to second order as flow goes to infinity.

$$b = 2\delta \tag{5}$$

$$c = (ln\beta + 1)\delta^2 - a^2/2 \tag{6}$$

where

$$\alpha = \frac{2.51}{4}\pi\nu\phi \tag{7}$$

$$\beta = \frac{k}{3.71\phi} \tag{8}$$

$$\delta = \frac{2\alpha}{\beta ln10} \tag{9}$$

It is noted that the value of c depends on the value of a which is chosen subsequently.

The parameters a and d are open to selection and may be chosen to match any desired slope at zero flow (Burgschweiger et al. 2009). However, choosing the slope at zero flow or parameters a and d is not discussed in detail beyond requiring $b > 0$ and $c < 0$ along with $a > 0$ and $d > 0$.

SELECTING OPEN PARAMETERS

Finding values of the parameters a and d to yield a chosen slope at zero flow requires solution of an under-determined equation. At zero flow, the slope of the smoothed friction loss curve according to Eq 3 is

$$\hat{h}'(0) = r(a + b + c/d). \tag{10}$$

Substituting for r, b and c in Eq 10 and equating with a desired slope h_0' yields one equation with two unknowns. Solutions may be found applying the method of least squares (Kreyszig 1999).

Define Δ as the difference between Eq 10 and the target slope h_0' and substitute for c.

$$\Delta = ra + rb + \frac{r}{d}(\ln\beta + 1)\delta^2 - \frac{ra^2}{2d} - h_0' \tag{11}$$

Define $F = \Delta^2$ as the squared difference between the slopes. Taking the partial derivative of F with respect to a and equating to zero gives

$$\frac{\partial F}{\partial a} = 2\Delta r(1 - a/d) = 0 \tag{12}$$

from which two conditions are available: $\Delta = 0$ and $a = d$. Taking the partial with respect to d gives

$$\frac{\partial F}{\partial d} = -2\Delta r \frac{c}{d^2} = 0 \tag{13}$$

and again two conditions are available: $\Delta = 0$ and $c = 0$. Choosing $c = 0$ would destroy the asymptotic characteristics of the approximation. Values for a and d are thus chosen by setting $d = a$ in Eq 11 and solving for a using the quadratic formula.

$$a = \frac{\left(h_0^{'} - rb\right) \pm \sqrt{(rb - h_0^{'}) - 2r^2(ln\beta + 1)\delta^2}}{r} \tag{14}$$

Between the two possible values of a from Eq 14 the smaller value which satisfies $a > 0$ is chosen.

Very near an average flow rate of zero, Hagen-Pousille flow occurs and the Darcy friction factor is $f = 64/Re$ (White 1999). The slope of the friction loss curve at zero is given by

$$h_0^{'} = \frac{128\nu L}{\pi d^4 g} \tag{15}$$

Using Eq 15 with Eq 14 yields a value for a and d.

Theoretical and approximate head loss curves are shown for several roughness values in Figs 1 and 2. In both figures, the example pipes have L=1000m, ϕ=.2 m, and $\nu = 1.00510^{-6}\frac{m^2}{s}$ for water at 20 C. In Figure 1, the slope of the theoretical and approximate curves near zero flow show that the derived expressions provide the desired slope. In the laminar ($Re < 2300$) regime, the approximation over-estimates head loss while in the turbulent regime, losses are under-estimated. The rougher pipe, being nearer the limiting case of Eq 4 is approximated more closely than the smoother pipe.

In Figure 2 head loss curves are shown for a wider range of Reynolds number. The error between theoretical and approximate curves visible in Figure 1 for $k/\phi = 0.001$ is not visible at higher Reynolds number, suggesting the approximation converged to the theoretical curve. In contrast, a smoother pipe having relative roughness 0.00005 does not exhibit convergence of the approximation toward the theoretical curve one in the range shown. The error incurred by the smoothed model depends on the relative roughness of the pipe and the Reynolds number of the flow in question. A more detailed view of the errors is presented in the next section.

ACCURACY ASSESSMENT

The accuracy of the approximation was assessed over a grid of relative roughness (k/ϕ) and Reynolds number (Re). This study used a grid size of 200x200 with points spaced logarithmically over the ranges: $10^{-6} < \frac{d}{\phi} < 0.05$ and $4000 < Re < 10^8$ considered on the diagram of Moody (1944). Computations were carried out in the R environment (version 3.0.2, (R Core Team 2013)). Theoretical values were obtained by applying Eqs 1 and 2. Inversion of the latter used used a bisection procedure combined with interpolation as implemented in the uniroot() function of R with tolerance 10^{-10}.

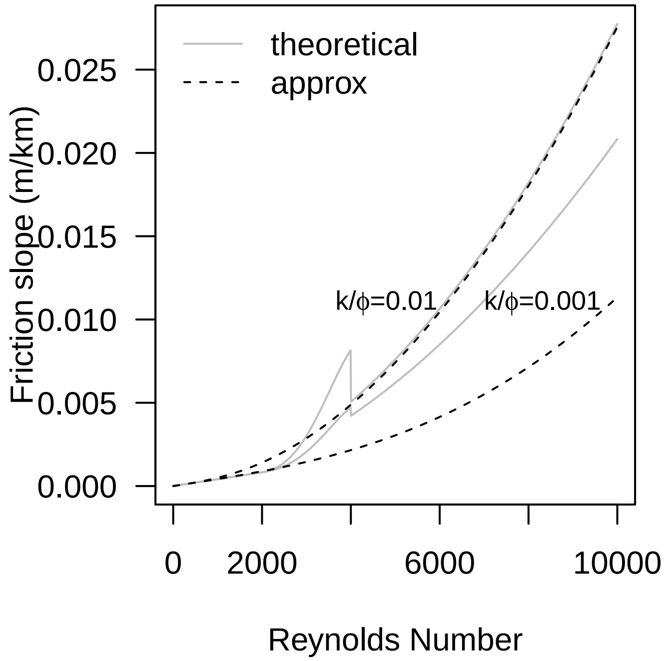

Figure 1. Smoothed model compared to theoretical values at lower Reynolds numbers. Slope at zero is found from laminar conditions.

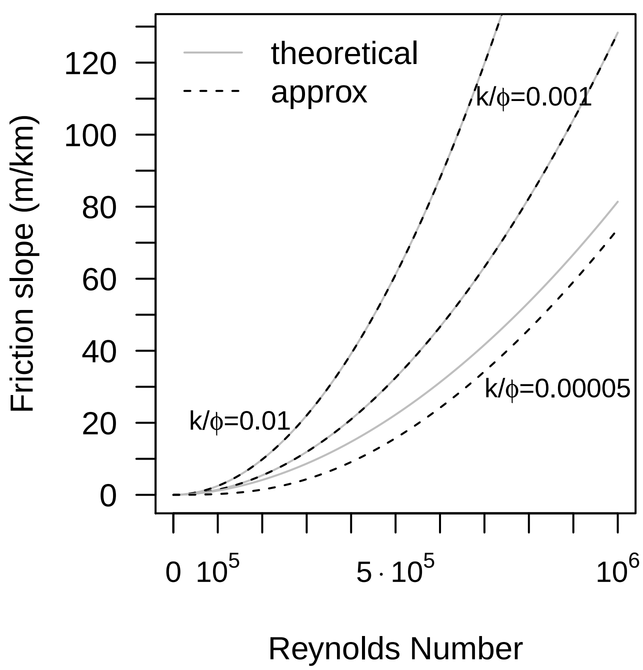

Figure 2. Smoothed model compared to theoretical values at higher Reynolds numbers. Slope at zero is found from laminar conditions.

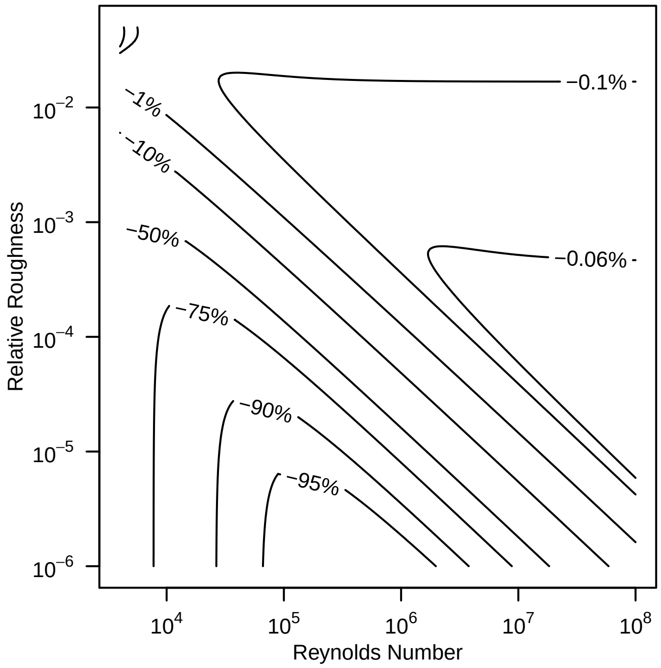

Figure 3. Relative errors of smoothed approximation with Hagen-Pouiselle slope at zero flow considering water at 20C.

Over the chosen grid relative errors ranged from -98% to -0.035%. A contour plot (Fig. 3) shows errors larger error for smoother pipes. The asymptotic behavior designed into the model is observed as errors tending to zero as Reynold's number increases. However, for smoother pipes higher and higher Reynolds numbers are needed to observe the small errors. The plot may be used to assess the suitability of the approximation for different situations. For example, allowing an under-estimate in head loss of 10% in the $Re > 4000$ flow range, pipes should have a relative roughness of .007 or higher. With an internal diameter of 300 mm this corresponds to a roughness height of 2.1 mm.

CONCLUSIONS

This note has provided formulae for setting the two free coefficients of Burgschweiger's smoothed approximation based on a desired slope at zero flow. Using a least squares approach, equating the two coefficients are shown to achieve the desired slope. Laminar flow conditions give a closed form expression for the final parameter. Setting the open parameters to achieve a slope at zero flow corresponding to laminar conditions under-estimated head losses. The approximation accuracy increases with Reynold's number and relative roughness.

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